Lecture Notes of C2.6 Introduction to schemes.*

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Contents

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0 Introduction

This is a first technical introduction to the theory of schemes.

The precise prerequisites are:

- Chapters I, II and III (ie pp. 1–50) of [AM69] (basic definitions of commutative algebra).
- Appendix A (ie pp. 417–432) of [Wei94] (basic definitions of category theory).
- Section 1, 2 and 3 of Chapter I (ie pp. 1–15) of [Wei94] (basic definitions of homological algebra).
- The first section of Chapter 3, par. 5 (ie pp. 438–443) of [GH94] (the cohomological spectral sequence of a double complex).
- The definition of a topological space.

A first course in algebraic geometry is desirable but not technically necessary.

The terminology of these notes will generally be in line with the terminology of Hartshorne's book [Har77].

The theory of schemes was developed by A. Grothendieck and his school, in an attempt to give an intrinsic description of the objects of algebraic geometry, as opposed to the classical extrinsic description in terms of varieties, which always come with an embedding into affine or projective space.

For details and lucid explanations on the material described in these notes, the reader is strongly encouraged to consult the foundational treatise [GD]. The Stacks Project, which can be accessed at the web address

`http://stacks.math.columbia.edu/`

contains detailed presentations of all the concepts introduced in [GD] and its later ramifications.

Other valuable references are [Liu02], [GW10] and also the online notes of Ravi Vakil at Stanford university. See

`http://math.stanford.edu/˜vakil/216blog/FOAGjun1113public.pdf`

for the latter.

The books [Mum99] and [EH00] can serve as conceptual introductions to the subject.

The structure of these notes of course reflect the author's biases. Here are some of them:

- *Homological algebra.* The language and basic results of homological algebra are prerequisites to this text and are not introduced alongside the main geometrical results presented in this text. Our feeling is that introducing tools from homological algebra alongside geometrical constructions diverts the student's attention from the real difficulties.
- *Cohomology.* Cohomology appears at the very beginning of this text. Algebraic geometry extends commutative algebra to a global situation and cohomology is a systematic procedure for dealing with the interaction between local and global. It thus makes sense to use it from the very beginning.
- *Category theory.* We express many facts in categorical terms and we use categorical arguments (esp. involving adjoint functors) when possible. We thus avoid many redundancies.
- *Moduli.* We insist on the duality between schemes and the corresponding functor of points (ie the associated contravariant functor in the category of schemes). Many (most?) fundamental objects in algebraic geometry represent certain important moduli problems and should be viewed as such. This applies in particular to affine and projective space.

Caveat emptor. These notes are very terse. They are technically self-contained but are difficult to follow if not supplemented by other sources.

Conventions. In this text, unless explicitly stated otherwise, all rings are assumed to be commutative and unital.

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Further reading

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1 Sheaves on topological spaces and their cohomology

1.1 Cohomology

Let $\mathcal{A}$ be an abelian category. An object I of $\mathcal{A}$ is called *injective* if the contravariant functor

$$\operatorname{Hom}_{\mathcal{A}}(\bullet, I) : \mathcal{A} \to \mathbf{Ab}$$

is exact.

Let $A^\bullet$ be a cochain complex in $\mathcal{A}$, which is bounded below. An *injective resolution* of $A^\bullet$ is a cochain complex in A

$$I^\bullet : \ldots \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \ldots$$

such that:

- for all $k \in \mathbb{Z}$, the object I^k is injective;
- $I^\bullet$ is bounded below;
- there is a morphism of complex $A^\bullet \to I^\bullet$, which is a quasi-isomorphism.

If every object in $\mathcal{A}$ has an injective resolution, we say that $\mathcal{A}$ *has enough injectives.* Notice that this is equivalent to requiring that for any object A in $\mathcal{A}$ there is a monomorphism of A into an injective object. In that case every cochain complex $A^\bullet$ in $\mathcal{A}$, which is bounded below, has an injective resolution. See [Har66, I.4.6i] for the proof (see also the proof of Theorem 1.3 below).

Let $(B^\bullet, d_B^\bullet)$ and $(C^\bullet, d_C^\bullet)$ be cochain complexes in $\mathcal{A}$ and let $f^\bullet, g^\bullet : B^\bullet \to C^\bullet$ be two morphisms of complexes. A *homotopy* $k^\bullet$ between $f^\bullet$ and $g^\bullet$ is a collection of morphisms $k^i : B^i \to C^{i-1}$ $(i \in \mathbb{Z})$ such that $f^i - g^i = d_C^{i-1} \circ k^i + k^{i+1} \circ d^i$ for all $i \in \mathbb{Z}$.

Lemma 1.1. *The relation 'existence of a homotopy between two complexes' is an equivalence relation. If $f^\bullet$ and $g^\bullet$ as above are homotopic then $\mathcal{H}^k(f^\bullet) = \mathcal{H}^k(g^\bullet)$ for all $k \in \mathbb{Z}$, ie $f^\bullet$ and $g^\bullet$ induce the same morphisms in homology.*

Proof. This is exercise 1.3. □

Lemma 1.2. *Let $\phi : A \to B$ be a morphism of objects of $\mathcal{A}$. Let $I^\bullet$ (resp. $J^\bullet$) be an injective resolution of A (resp. B). Then there is a morphism of complexes $I^\bullet \to J^\bullet$, which is compatible with the morphisms $A \to I^0$, $B \to J^0$ and ϕ. Any two such morphisms are homotopic.*

Here we view an object of $\mathcal{A}$ as a cochain complex concentrated in the index 0.

Proof. Exercise 1.4. □

Let $\mathcal{B}$ be another abelian category.

Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor. We say that F is *additive* if for all objects A, B of $\mathcal{A}$, the map $\operatorname{Mor}(A, B) \to \operatorname{Mor}(F(A), F(B))$ is a map of abelian groups.

We say that F is *left exact* if for any exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to F(A') \to F(A) \to F(A'')$$

is also exact.

Suppose that $\mathcal{A}$ has enough injectives.

If $F : \mathcal{A} \to \mathcal{B}$ is a covariant (not necessarily left exact) additive functor, we may for all $i \in \mathbb{Z}$ define a functor R^iF by the following recipe. For A and object in $\mathcal{A}$, let $I^\bullet$ be a injective resolution of A (here A is viewed as a cochain complex concentrated in the index 0). Now define $R^iF(A) := \mathcal{H}^i(F(I^\bullet))$. By Lemma 1.2, the object $\mathcal{H}^i(F(I)^\bullet)$ is well-defined up to unique isomorphism. Furthermore, the same lemma shows that associated with any morphism $A \to B$ in $\mathcal{A}$, there is a canonical morphism $R^iF(A) \to R^iF(B)$, and that this association makes R^iF into an additive functor. The functor R^iF is the i-th *right derived* functor of F. By construction, there is a natural isomorphism of functors $R^0F \simeq F$.

The range of the functor R^iF can be extended to all complexes $A^\bullet$ in $\mathcal{A}$, which are bounded below. One then similarly defines

$$R^i(A^\bullet) := \mathcal{H}^i(F(I^\bullet))$$

where $I^\bullet$ is an injective resolution of $A^\bullet$. The object $\mathcal{H}^i(F(I^\bullet))$ is then well-defined up to unique isomorphism.

Theorem 1.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor to another abelian category. For any short exact sequence*

$$0 \to A' \to A \to A'' \to 0 \tag{1}$$

there is a canonical 'long' exact sequence

$$0 \to R^0F(A') \to R^0(A) \to R^0F(A'') \to R^1F(A') \to R^1(A) \to R^1F(A'') \to \dots \tag{2}$$

In this sequence, the subsequence $R^iF(A') \to R^i(A) \to R^iF(A'')$ is the image of the sequence $A' \to A \to A''$ by R^iF. Furthermore, the sequence (2) is naturally functorial in the short exact sequence (1).

Proof. Sketch. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^{',1} & \to & I^1 & \to & I^{'',1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^{',0} & \to & I^0 & \to & I^{'',0} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & A' & \to & A & \to & A'' & \to & 0
\end{array}$$

where the vertical sequences are injective resolutions (see [Lan02, Lemma 9.5]) for the details). The theorem follows from this and the existence of the long exact cohomology sequence associated with an exact sequence of complexes (see [Wei94, Th. I.3.1]). □

Theorem 1.3 can be generalised to general complexes as follows:

Theorem 1.4. *Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor to another abelian category. Let $A^\bullet$ be a cochain complex in $\mathcal{A}$, which is bounded below. Then there is a E_1 cohomological spectral sequence*

$$E_1^{pq} = R^q F(A^p) \Rightarrow R^{p+q} F(A^\bullet)$$

where the the morphisms $R^q F(A^p) \to R^q F(A^{p+1})$ are induced by the morphisms $A^p \to A^{p+1}$ for all $p, q \in \mathbb{Z}$. There is also an E_2 cohomological spectral sequence

$$E_2^{pq} = R^p F(\mathcal{H}^q(A^\bullet)) \Rightarrow R^{p+q} F(A^\bullet)$$

Both spectral sequences are naturally functorial in $A^\bullet$.

Proof. Sketch. The proof is similar to the proof Theorem 1.3, with an added twist. As before, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & A^{0} & \longrightarrow & A^{1} & \longrightarrow & A^{2} & \longrightarrow & \dots
\end{array}$$

where the $I^{i,j}$ are injective objects. Now it is possible to construct this diagram in such a way that for each $k \in \mathbb{Z}$, the corresponding sequence of homology objects

$$0 \to \mathcal{H}^k(A^\bullet) \to \mathcal{H}^k(I^{0,\bullet}) \to \mathcal{H}^k(I^{1,\bullet}) \to \dots$$

is exact and the homology objects $\mathcal{H}^k(I^{r,k})$ are injective for all $r, k \in \mathbb{Z}$. See [Lan02, XX, par. 9, Lemma 9.5] for this. The two spectral sequences are then nothing but the two spectral sequences of the double complex associated with $I^{\bullet,\bullet}$. □

Theorem 1.4 will only be used Exercise 1.5 below. The reader who wants to postpone learning and using spectral sequences can take Exercise 1.5 for granted and skip Theorem 1.4.

Remark 1.5. The theory of derived functors was developed in Grothendieck's article [Gro57], which is still the most valuable reference for the material of this section.

1.2 Sheaves

Let X be a topological space. Denote the category of abelian groups by **Ab**. We define $\mathrm{Top}(X)$ as the category whose object are the open sets of X and whose arrows (=morphisms) are the inclusion maps.

Definition 1.6. *A presheaf F (of abelian groups) on X is a contravariant functor $F : \mathrm{Top}(X) \to$ **Ab**.*

The presheaves on X naturally form a category, whose arrows are the natural transformations of functors. If $U \to V$ are is an inclusion of open subsets of X and $s \in F(V)$, we write

$$s|_U := F(U \to V)(s).$$

A *sheaf* on X is a presheaf F on X, with the following properties. Let $(U_i \in \mathrm{Top}(X))$ be a family of open subsets of X. Then

- if for all indices i we are given $s_i \in F(U_i)$ and furthermore $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j then there is a unique element $s \in F(\bigcup_i U_i)$ such that $s|_{U_i} = s_i$ for all i.

Definition 1.7. *Let F be a presheaf on a topological space X. Let $x \in X$. The stalk F_x is*

$$F_x := \varinjlim_{U \in \mathrm{Top}(X),\ x \in U} F(U)$$

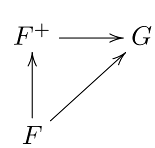
See [Wei94, Appendix A.5] for the notion of direct limit. Here is a direct construction of F_x. Let $\widetilde{F_x} := \coprod_{U \in \mathrm{Top}(X),\ x \in U} F(U)$. Here $\coprod$ refers to disjoint union. Define a relation $\sim_x$ on $\widetilde{F_x}$ by the following recipe. If U, V are open subsets of X containing x and

$$s \in F(U), t \in F(V),$$

then $s \sim_x t$ iff there is an open set $W \subseteq U \cap V$ containing x such that $s|_W = t|_W$. This relation is an equivalence relation and the set F_x is naturally isomorphic to the quotient of the set $\widetilde{F_x}$ by the relation $\sim_x$. This quotient has a unique structure of abelian group, such that for all $U \in \mathrm{Top}(X)$ such that $x \in U$, the natural map $F(U) \to F_x$ is a morphism of abelian groups. The reader is asked to provide the details in Exercise 1.6.

If $x \in X$ and $U \in \mathrm{Top}(X)$ contains x then for any $s \in F(U)$, we write s_x for the image of s in F_x. If $\phi : F \to G$ is a morphism of presheaves on X, there is a unique map of abelian groups $\phi_x : F_x \to G_x$ such that for any $s \in F(U)$ and $U \in \mathrm{Top}(X)$ containing x, we have $\phi_x(s_x) = (\phi(s))_x$.

Proposition-Definition 1.8. *Let F be a presheaf on a topological space X. There is sheaf F^+ on X and a natural transformation $F \to F^+$, both of which are uniquely defined up to unique isomorphism by the following property: if G is a sheaf on X and $F \to G$ is a natural transformation, then there is a* **unique** *natural transformation $F^+ \to G$ such that the diagram*



commutes. The sheaf F^+ is called the sheafification of F.

Proof. Let $E := \coprod_{x \in X} F_x$, where $\coprod$ refers to disjoint union. For any $U \in \mathrm{Top}(X)$, define

$$F^+(U) := \{f : U \to E \mid \text{for all } x \in U \text{ we have } f(x) \in F_x$$
$$\text{and for all } x \in U \text{ there is } U(x) \in \mathrm{Top}(U) \text{ containing } x$$
$$\text{and } s \in F(U(x)) \text{ such that } s_y = f(y) \text{ for all } y \in U(x)\}$$

It follows from the construction that F^+ is a sheaf. The natural transformation $F \to F^+$ is given for all by the formula

$$s \in F(U) \;\mapsto \text{function } f : U \to E \text{ such that } f(x) = s_x \text{ for all } x \in U,$$

which is valid for all $U \in \mathrm{Top}(X)$. If G is a sheaf and $F \to G$ is a natural transformation, then we may define a natural transformation $F^+ \to G$ by the following recipe. Let $U \in \mathrm{Top}(X)$. We use the symbols appearing in the definition of F^+. We wish to associate an element of $G(U)$ with $f : U \to E$. To this end consider the covering of U given by the U_x appearing in the definition of F^+. With each $s(x) \in F(U_x)$ we associate $t(x) \in G(U_x)$ via the natural transformation $F \to G$. The elements $t(x)$ verify the sheaf property because the $s(x)$ do. Hence they glue to a unique element of $G(U)$, because G is a sheaf. This is the image of f in $G(U)$. We leave it to the reader to verify that this defines a natural transformation $F^+ \to G$ and that it is the only one making the diagram in Proposition-Definition 1.8 commute. □

Remark 1.9. Let X be a topological space and let $\mathcal{C}$ be a category. The notion of sheaf and of sheafification makes sense if we define more generally a presheaf as a functor from $\mathrm{Top}(X)$ to $\mathcal{C}$ (not necessarily the category of abelian groups). We shall sometimes use the notion of sheaf in this generality in this text but we shall then speak of presheaves or sheaves *with values in* $\mathcal{C}$.

If $\phi : F \to G$ is a morphism of sheaves (=natural transformation of functors) on a topological space X, then we define the *kernel* $\ker(\phi)$ of ϕ as the presheaf

$$U \in \mathrm{Top}(X) \;\mapsto\; \ker(\phi(U))$$

This presheaf is a sheaf: see Exercise 1.7.

We define the *cokernel* $\mathrm{coker}(\phi)$ of ϕ as the sheafification of the presheaf

$$U \in \mathrm{Top}(X) \;\mapsto\; \mathrm{coker}(\phi(U))$$

Proposition-Definition 1.10. *Let X be a topological space. The category $\mathbf{Ab}(X)$ of sheaves on X is an abelian category. If $\phi : F \to G$ is a morphism of sheaves, then the categorical kernel (resp. cokernel) of ϕ is canonically isomorphic to $\ker(\phi)$ (resp. $\mathrm{coker}(\phi)$). A cochain complex*

$$\cdots \to F^{i-1} \to F^i \to F^{i+1} \to \dots$$

is exact in $\mathbf{Ab}(X)$ if and only for any $x \in X$, the corresponding sequence of stalks

$$\cdots \to F_x^{i-1} \to F_x^i \to F_x^{i+1} \to \dots$$

is exact.

Proof. See Exercise 1.8. □

Let $f : X \to Y$ be a continuous map of topological spaces. For F is a sheaf on X, we define the presheaf $f_*(F)$ by the formula

$$V \in \mathrm{Top}(Y) \ \mapsto \ F(f^{-1}(V))$$

The presheaf $f_*(F)$ is a sheaf (easy). This construction naturally gives rise to an additive functor $\mathbf{Ab}(X) \to \mathbf{Ab}(Y)$, also denoted by f_*.

For F a sheaf on Y, we define the sheaf $f^{-1}(F)$ as the sheafification of the presheaf on X given by the formula

$$U \in \mathrm{Top}(X) \ \mapsto \ \varinjlim_{V \in \mathrm{Top}(Y), V \supseteq f(U)} F(V)$$

Again, this leads to an additive functor $\mathbf{Ab}(Y) \to \mathbf{Ab}(X)$, denoted by f^{-1}.

Proposition 1.11. *The functor f^{-1} is left adjoint to the functor f_*.*

Proof. We have to show that for any sheaf F on Y and any sheaf G on X, there is a natural isomorphism

$$\mathrm{Mor}(f^{-1}(F), G) \simeq \mathrm{Mor}(F, f_*(G))$$

and that this morphism is natural for morphisms of sheaves on Y (resp. on X). An element of $\mathrm{Mor}(f^{-1}(F), G)$ is a collection of maps of abelian groups

$$\varinjlim_{V_0 \supseteq f(U)} F(V_0) \to G(U)$$

(where $U \in \mathrm{Top}(X)$) satisfying certain compatibility properties. An element of $\mathrm{Mor}(F, f_*(G))$ is a collection of maps of abelian groups

$$F(V) \to G(f^{-1}(V))$$

(where $V \in \mathrm{Top}(Y)$) again satisfying certain compatibility properties. We wish to establish a bijective correspondence between $\mathrm{Mor}(f^{-1}(F), G)$ and $\mathrm{Mor}(F, f_*(G))$. Start with $\mathrm{Mor}(f^{-1}F, G)$. Setting $U := f^{-1}(V)$, we obtain a map

$$\varinjlim_{V_0 \supseteq f(f^{-1}(V))} F(V_0) \to G(f^{-1}(V)) \tag{3}$$

Now since $V \supseteq f(f^{-1}(V))$, we have natural map $F(V) \to \varinjlim_{V_0 \supseteq f(f^{-1}(V))} F(V_0)$. Composing this map with the map (3), we obtain a map

$$F(V) \to G(f^{-1}(V))$$

We leave to the reader to show that this indeed defines a natural transformation of functors $\mathrm{Top}(Y) \to \mathbf{Ab}$.

Now start with $\mathrm{Mor}(F, f_*(G))$. Applying $\varinjlim$, we obtain a map

$$\varinjlim_{V_0 \supseteq f(U)} F(V_0) \to \varinjlim_{V_0 \supseteq f(U)} G(f^{-1}(V_0)) \tag{4}$$

Now for any $V_0 \supseteq f(U)$, we have $f^{-1}(V_0) \supseteq U$ and hence there is a natural map

$$\varinjlim_{V_0 \supseteq f(U)} G(f^{-1}(V_0)) \to G(U).$$

Composing the map (4) with this map, we obtain a map

$$\varinjlim_{V_0 \subseteq f(U)} F(V_0) \to G(U)$$

We again leave it to the reader to show that this defines a natural transformation of functors. One can easily verify that the maps $\mathrm{Mor}(f^{-1}(F), G) \to \mathrm{Mor}(f^{-1}(F), G)$ and $\mathrm{Mor}(f^{-1}(F), G) \to \mathrm{Mor}(f^{-1}(F), G)$ that we have just described are inverse to each other and are natural in the sheaves F and G. $\square$

Remark 1.12. The fact that f^{-1} and f_* are adjoint to each other formally implies that f_* is left exact and that f^{-1} is right exact. See Exercise 1.2.

If (F_i) is a family of sheaves on a topological space X, we define the presheaf $\prod_i F_i$ by the formula

$$U \in \mathrm{Top}(X) \ \mapsto \ \prod_i F_i(U)$$

where $\prod_i F_i(U)$ is the product of the abelian groups $F_i(U)$ (ie the cartesian product of the sets $F_i(U)$, endowed with the evident group structure). It can easily be verified that the presheaf $\prod_i F_i$ is a sheaf. By construction, if G is another sheaf on X, we have an identification

$$\mathrm{Mor}(G, \prod_i F_i) \simeq \prod_i \mathrm{Mor}(G, F_i)$$

Theorem 1.13. *Let X be a topological space. The category* $\mathbf{Ab}(X)$ *has enough injectives.*

Proof. We shall use the fact that **Ab** is a category with enough injectives. See Exercise 1.10 for this. Let F be a sheaf on X. We shall construct an injective sheaf I and a monomorphism $F \to I$. For each $x \in X$, choose an injective abelian group I_x and an injection $\iota_x : F_x \to I_x$. Denote also by x the inclusion map $x \to X$, where x is viewed as a topological space. Define

$$I := \prod_{x \in X} x_*(I_x)$$

Note that by construction we have for all $U \in \mathrm{Top}(X)$ an isomorphism

$$I(U) \simeq \prod_{x \in U} I_x$$

which is compatible with restrictions to smaller open sets. In particular, we may define a morphism $F \to I$ by the formula

$$s \in F(U) \ \mapsto \ \prod_{x \in U} \iota_x(s_x)$$

This morphism is a monomorphism: if the image of $s \in F(U)$ vanishes, then $s_x = 0$ for all $x \in U$; hence $s = 0$ by the sheaf property.

Now let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of sheaves on X. We wish to show that the corresponding sequence

$$0 \to \mathrm{Mor}(F'', I) \to \mathrm{Mor}(F, I) \to \mathrm{Mor}(F', I) \to 0 \tag{5}$$

is exact. Now we have natural isomorphisms

$$\mathrm{Mor}(F, I) \simeq \mathrm{Mor}(F, \prod_{x \in X} x_*(I_x)) \simeq \prod_{x \in X} \mathrm{Mor}(F, x_*(I_x)) \simeq \prod_{x \in X} \mathrm{Mor}(x^{-1}(F), I_x) \simeq \prod_{x \in X} \mathrm{Mor}(F_x, I_x).$$

For the second isomorphism, we used the remarks before the statement of this theorem. For the third isomorphism, we used Proposition 1.11 and for the last isomorphism, we used the fact that by definition $x^{-1}(F) = F_x$. Hence the sequence (5) is isomorphic to the product over all $x \in X$ of the sequences

$$0 \to \mathrm{Mor}(F''_x, I_x) \to \mathrm{Mor}(F_x, I_x) \to \mathrm{Mor}(F'_x, I_x) \to 0$$

which are exact because the I_x are injective abelian groups and the sequences

$$0 \to F'_x \to F_x \to F''_x \to 0$$

are exact by Proposition 1.10. □

1.3 Cohomology of sheaves

The functor

$$\Gamma(X, \bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$$

described by the formula

$$\Gamma(X, F) := F(X)$$

is left exact. We shall often write $H^i(X, \bullet)$ for the i-th right derived functor $R^i\Gamma(X, \bullet)$ of $\Gamma(X, \bullet)$.

More generally, let $f : X \to Y$ be a continuous map of topological spaces. The functor

$$f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$$

is left exact. This is a consequence of Remark 1.12 but we ask to reader to give a direct proof in Exercise 1.9. We shall record an important consequence of the proof of Theorem 1.13 in the following

Proposition 1.14. *Let $f : X \to Y$ be a continuous map of topological spaces and let $V \in \mathrm{Top}(Y)$. Let $U := f^{-1}(V)$ and let $u : U \to X$, $v : V \to Y$ be the inclusion maps. Let $f_V : U \to V$ be the natural map. Let F be a sheaf on X. For all $i \geqslant 0$, we have canonical isomorphisms*

$$v^{-1}(R^i f_*(F)) \simeq R^i f_{V,*}(u^{-1}(F)).$$

and these isomorphisms are natural in F.

Proof. First notice that the theorem holds for $i = 0$, by the definition of f_*. To compute the various derived functors, we choose an injective resolution $I^\bullet$ for F. We thus have an exact sequence

$$0 \to F \to I_0 \to I_1 \to \dots$$

We may choose all the I^k as in the proof of Theorem 1.13. These injective sheaves have the property that for any open set $u_0 : U_0 \to X$ of X, the sheaf $u_0^{-1}(I^k)$ is injective on U_0. Now we may compute

$$v^{-1}(R^i f_*(F)) \simeq v^{-1}(\mathcal{H}^i(f_*(I^\bullet))) \simeq \mathcal{H}^i(u^{-1}(f_*(I^\bullet))) \simeq \mathcal{H}^i(f_{U,*}(v^{-1}(I^\bullet))) \simeq R^i f_{U,*}(u^{-1}(F))$$

The naturality in F follows from Lemma 1.2. □

Cech cohomology. We shall now describe how a sheaf can be related categorically to its restrictions to open subsets. So let F be a sheaf on a topological space X. Let I be a finite set and let $(U_{i \in I})$ be a covering of X by open sets indexed by I. In the following discussion, when $i_0, \dots i_p \in I$, we shall use the short-hand $i_0 \dots i_p$ for $(i_0, \dots, i_p) \in I^{\{0,\dots,p\}}$. For $i_0, \dots i_p \in I$, we define

$$U_{i_0 \dots i_p} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p}$$

and we let $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ be the inclusion map. Furthermore, for all $p \geqslant 0$, let

$$\underline{C}^p((U_i), F) := \bigoplus_{i_0 \ldots i_p} j_{i_0 \ldots i_p,*}(j^{-1}_{i_0 \ldots i_p}(F))$$

We now define a morphism of sheaves

$$d^p : \underline{C}^p((U_i), F) \to \underline{C}^{p+1}((U_i), F)$$

by the formula

$$\bigoplus_{i_0 \ldots i_p} \alpha_{i_0 \ldots i_p} \mapsto \sum_{k=0}^{p+1} (-1)^k \bigoplus_{i_0 \ldots i_{p+1}} \alpha_{i_0 \ldots \hat{i_k} \ldots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}} \cap V}$$

where $V \in \mathrm{Top}(X)$ and

$$\alpha_{i_0 \ldots i_p} \in j_{i_0 \ldots i_p,*}(j^{-1}_{i_0 \ldots i_p}(F))(V) = F(U_{i_0 \ldots i_p} \cap V)$$

The hat symbol $\hat{}$ signifies that the term under the hat is omitted. Furthermore, we define a morphism

$$d : F \to \underline{C}^0((U_i), F) = \bigoplus_i j_{i,*} j_i^{-1}(F)$$

by taking the direct sum of the natural morphisms

$$F \to j_{i,*} j_i^{-1}(F)$$

arising from Proposition 1.11.

Theorem 1.15. *The sequence of sheaves*

$$0 \to F \xrightarrow{d} \underline{C}^0((U_i), F) \xrightarrow{d^0} \underline{C}^1((U_i), F) \xrightarrow{d^1} \ldots \tag{6}$$

is an exact cochain complex.

Proof. We leave it to the reader to verify that (6) is a complex. Since (6) is a complex of sheaves, it is sufficient to prove that its restriction to any U is exact for any open set U contained in some U_l. If we write $j : U \to X$ for the inclusion map, the restriction of (6) to U is naturally isomorphic to the Cech complex for $j^{-1}(F)$ associated with the open covering $(U_{i \in I} \cap U)$ of U. Note that this open covering contains $U_l \cap U = U$. So it is sufficient to prove the theorem under the supplementary hypothesis that for one of the open sets U_i we have $U_i = X$.

So we suppose that $U_\rho = X$ until the end of the proof.

We shall need the following notation. If the image of the sequence $i_0 \ldots i_p$ is contained in the image of the sequence $l_0 \ldots l_q$, there is a inclusion map $U_{l_0 \ldots l_q} \to U_{i_0 \ldots i_p}$ that we shall denote by $j_{l_0 \ldots l_q \mapsto i_0 \ldots i_p}$. The maps $j_{l_0 \ldots l_q \mapsto i_0 \ldots i_p}$ satisfy obvious transitivity properties and the maps

$$j_{i_0 \ldots i_k \rho\ i_{k+1} \ldots i_p \mapsto i_0 \ldots \ldots i_p}$$

are isomorphisms for any $k \in \{0, \ldots, p\}$. In this proof, we shall write $j^{[-1]}_{i_0 \ldots i_k \rho\ i_{k+1} \ldots i_p \mapsto i_0 \ldots \ldots i_p}$ for the inverse of $j_{i_0 \ldots i_k \rho\ i_{k+1} \ldots i_p \mapsto i_0 \ldots \ldots i_p}$.

We shall show that the sequence of abelian groups

$$0 \to F(X) \xrightarrow{d} \underline{C}^0((U_i), F)(X) \xrightarrow{d_0} \underline{C}^1((U_i), F)(X) \xrightarrow{d_1} \ldots \tag{7}$$

induced by (6) is exact. This is sufficient to prove the theorem because it implies that a sequence analogous to (7) is exact when X is replaced by any of its open subsets. From this, we can deduce that the sequence

$$0 \to F \xrightarrow{d} \underline{C}^0((U_i), F) \xrightarrow{d^0} \underline{C}^0((U_i), F) \xrightarrow{d^1} \dots$$

is exact (use Exercise 1.11).

To prove that the sequence (7) is exact, define for all $p \geqslant 1$ a map

$$h^p : \underline{C}^p((U_i), F)(X) \to \underline{C}^{p-1}((U_i), F)(X)$$

by the formula

$$h^p(\bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p}) := \bigoplus_{i_0 \dots i_{p-1}} j^{[-1],-1}_{\rho\ i_0 \dots i_{p-1} \mapsto i_0 \dots i_{p-1}}(\alpha_{\rho\ i_0 \dots i_{p-1}})$$

We compute

$$\begin{aligned} d^{p-1}h^p(\bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p})) &= d^{p-1}(\bigoplus_{i_0 \dots i_{p-1}} j^{[-1],-1}_{\rho\ i_0 \dots i_{p-1} \mapsto i_0 \dots i_{p-1}}(\alpha_{\rho\ i_0 \dots i_{p-1}})) \\ = \sum_{k=0}^{p} (-1)^k & \bigoplus_{i_0 \dots i_p} j^{-1}_{i_0 \dots i_p \mapsto i_0 \dots \hat{i_k} \dots i_p}(j^{[-1],-1}_{\rho\ i_0 \dots \hat{i_k} \dots i_p \mapsto i_0 \dots \hat{i_k} \dots i_p}(\alpha_{\rho\ i_0 \dots \hat{i_k} \dots i_p}))) \end{aligned} \tag{8}$$

and

$$\begin{aligned} h^{p+1}d^p(\bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p})) &= h^{p+1}(\sum_{k=0}^{p+1}(-1)^k \bigoplus_{i_0 \dots i_{p+1}} j^{-1}_{i_0 \dots i_{p+1} \mapsto i_0 \dots \hat{i_k} \dots i_{p+1}}(\alpha_{i_0 \dots \hat{i_k} \dots i_{p+1}})) \\ = \sum_{k=0}^{p+1} (-1)^k h^{p+1} & (\bigoplus_{i_0 \dots i_{p+1}} j^{-1}_{i_0 \dots i_{p+1} \mapsto i_0 \dots \hat{i_k} \dots i_{p+1}}(\alpha_{i_0 \dots \hat{i_k} \dots i_{p+1}})) \\ = \sum_{k=0}^{p+1} (-1)^k & \bigoplus_{i_0 \dots i_p} j^{[-1],-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p}(j^{-1}_{\rho\ i_0 \dots i_p \mapsto \rho\ i_0 \dots \widehat{i_{k-1}} \dots i_p}(\alpha_{\rho\ i_0 \dots \widehat{i_{k-1}} \dots i_p})) \\ = \sum_{k=0}^{p} (-1)^{k+1} & \bigoplus_{i_0 \dots i_p} j^{[-1],-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p}(j^{-1}_{\rho\ i_0 \dots i_p \mapsto \rho\ i_0 \dots \hat{i_k} \dots i_p}(\alpha_{\rho\ i_0 \dots \hat{i_k} \dots i_p})) \\ + \bigoplus_{i_0 \dots i_p} & j^{[-1],-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p}(j^{-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p}(\alpha_{i_0 \dots i_p})) \end{aligned} \tag{9}$$

In the third and fourth line, in the situation where $k = 0$, the symbol $\widehat{i_{k-1}}$ means that ρ is omitted. Now using the transitivity relations, we see that

$$j^{[-1],-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p} \circ j^{-1}_{\rho\ i_0 \dots i_p \mapsto \rho\ i_0 \dots \hat{i_k} \dots i_p} = j^{-1}_{i_0 \dots i_p \mapsto i_0 \dots \hat{i_k} \dots i_p} \circ j^{[-1],-1}_{\rho\ i_0 \dots \hat{i_k} \dots i_p \mapsto i_0 \dots \hat{i_k} \dots i_p}$$

and of course we have

$$j^{[-1],-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p} \circ j^{-1}_{\rho\ i_0 \dots i_p \mapsto i_0 \dots i_p} = \mathrm{Id}$$

Now if we use these identities while summing the expressions (8) and (9), we get

$$(d^{p-1}h^p + h^{p+1}d^p)(\bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p}) = \bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p}$$

This identity immediately implies that the $\mathcal{H}^p(\underline{C}^\bullet((U_i), F)(X)) = 0$. On the other hand, the kernel of d^0 in (7) is precisely the image of $F(X)$ by the sheaf property. We have thus shown that the sequence (7) is exact, which completes the proof of the theorem. □

Complement 1.16. A variant of the Cech complex that will be useful is the following. Choose a total ordering on the index set I. Define the complex $\underline{C}^\bullet((U_i), F)$ as before but restrict the sequences of indices $i_0 \dots i_p$ to strictly increasing sequences. We shall call this new complex the *Cech complex with ordering*. The Cech complex with ordering is naturally a subcomplex of the original Cech complex and Theorem 1.15 is still valid for this complex. To see this, modify the definition of the map h^p in the proof of Theorem 1.15 as follows. Define as before

$$h^p(\bigoplus_{i_0\dots i_p} \alpha_{i_0\dots i_p}) := \bigoplus_{i_0\dots i_{p-1}} j^{[-1],-1}_{\rho\ i_0\dots i_{p-1}\mapsto i_0\dots i_{p-1}}(\alpha_{\rho\ i_0\dots i_{p-1}})$$

with the following conventions. If the sequence $\rho\ i_0 \dots i_{p-1}$ is not strictly increasing but is injective (ie there are no repetitions of indices) then let σ is the unique permutation of $\{\rho,\ i_0, \dots, i_{p-1}\}$ such that $\sigma(\rho)\ \sigma(i_0) \dots \sigma(i_{p-1})$ is increasing. We then define

$$\alpha_{\rho\ i_0\dots i_{p-1}} := (-1)^{\mathrm{sign}(\sigma)} \cdot \alpha_{\sigma(\rho)\ \sigma(i_0)\dots\sigma(i_{p-1})}$$

and

$$j^{[-1],-1}_{\rho\ i_0\dots i_{p-1}\mapsto i_0\dots i_{p-1}}(\alpha_{\rho\ i_0\dots i_{p-1}}) := j^{[-1],-1}_{\sigma(\rho)\ \sigma(i_0)\dots\sigma(i_{p-1})\mapsto i_0\dots i_{p-1}}(\alpha_{\sigma(\rho)\ \sigma(i_0)\dots\sigma(i_{p-1})})$$

Here $\mathrm{sign}(\sigma)$ is the sign of σ. If the sequence $\rho\ i_0 \dots i_{p-1}$ is not injective, then we set

$$\alpha_{\rho\ i_0\dots i_{p-1}} = j^{[-1],-1}_{\rho\ i_0\dots i_{p-1}\mapsto i_0\dots i_{p-1}}(\alpha_{\rho\ i_0\dots i_{p-1}}) = 0$$

The proof of Theorem 1.15 then goes through verbatim for the Cech complex with ordering in place of the Cech complex.

Note that the Cech complex with ordering only has a finite number of non-vanishing terms.

We now explain how to glue sheaves defined on open subsets.

Suppose given (U_i) an open covering of topological space X. If $j : U \to X$ is an open subset of X and F is a sheaf on X, we shall often write $F|_U$ instead of $j^{-1}(F)$.

Suppose given on U_i a sheaf F_i. Suppose given isomorphisms $\phi_{ij} : F_i|_{U_i\cap U_j} \xrightarrow{\sim} F_j|_{U_i\cap U_j}$ for all indices i, j, satisfying the properties (1), (2), (3) below.

(1) ϕ_{ii} is the identity;

(2) $\phi_{ji} = \phi_{ij}^{-1}$;

(3) $\phi_{ik}|_{U_i\cap U_j\cap U_k} = \phi_{jk} \circ \phi_{ij}|_{U_i\cap U_j\cap U_k}$.

for all indices i, j, k.

If F is a sheaf on X, the sheaves $F_i := F|_{U_i}$ come with the isomorphisms

$$\phi_{ij} : F_i|_{U_i\cap U_j} \xrightarrow{\sim} F_j|_{U_i\cap U_j}$$

coming from the natural isomorphism of functors $((\bullet)|_{U_i})_{U_i\cap U_j} \xrightarrow{=} ((\bullet)|_{U_j})_{U_i\cap U_j}$. We leave it to the reader to verify that the ϕ_{ij} verify (1), (2), (3) above.

The following proposition establishes a converse.

Proposition 1.17. *Given sheaves F_i on U_i and isomorphisms $\phi_{ij} : F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$ satisfying (1), (2), (3) above, there up to unique isomorphism a sheaf F on X with the following properties. There are isomorphisms $\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that the natural isomorphism*

$$(\psi_j^{-1}|_{U_i \cap U_j}) \circ \phi_{ij} \circ (\psi_i|_{U_i \cap U_j})$$

is the isomorphism

$$(F|_{U_i})_{U_i \cap U_j} \simeq (F|_{U_j})_{U_i \cap U_j}.$$

Proof. Let $E := \coprod_i E_i$, where $E_i := \coprod_{x \in U_i} F_{i,x}$. Notice that there is an obvious projection map $\pi : E \to X$. We define a relation $\sim$ on E in the following way. Let $e, f, g \in E$. If $e \in E_i$ and $f \in E_j$ lie above the same point $x \in U_i \cap U_j$, then we declare that $e \sim f$ if $f = \phi_{i,j,x}(e)$. Property (1) above shows that $e \sim e$. Property (2) shows that $f \sim e$ if $e \sim f$. Property (3) shows that if $e \sim f$ and $f \sim g$ then $e \sim g$. We have thus shown that $\sim$ is an equivalence relation.

We shall write $q : E \to E/\sim$ for the mapping of E to the quotient space of E by $\sim$ (=the family of equivalence classes of $\sim$). The quotient space $E/\sim$ comes with a natural map $\rho : (E/\sim) \to X$ such that $\rho \circ q = \pi$. This is because $\sim$ only identifies points of E, which lie over the same point in X. Furthermore the maps $q|_{E_i} : E_i \to E/\sim$ are by construction injective. We now define a sheaf F on X by the recipe

$$\begin{aligned} F(U) := \{ s : U \to E/\sim \ | \ \rho \circ s = \mathrm{Id}_U \text{ and for all } x \in U \text{ there is} \\ \text{an index } i, \text{ an open set } V(x) \in \mathrm{Top}(U_i) \text{ containing } x \text{ and an element } t \in F_i(V(x)) \\ \text{such that } s(v) = q_i|_{E_i}(t_v) \text{ for all } v \in V(x) \} \end{aligned}$$

for $U \in \mathrm{Top}(X)$.

By construction, $F|_{U_i}$ is canonically isomorphic to F_i for any index i. It can easily be verified that these canonical isomorphisms verify the requirements of the proposition. ☐

Complement 1.18. Let X be a topological space and (U_i) a covering of X by open subsets. Suppose given sheaves F_i (resp. G_i) on U_i and isomorphisms $\phi_{ij} : F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$ (resp. $\psi_{ij} : G_i|_{U_i \cap U_j} \xrightarrow{\sim} G_j|_{U_i \cap U_j}$) satisfying the properties (1), (2), (3) above. Let F (resp. G) be the sheaf associate with these data by Proposition 1.17. Then to give a morphism of sheaves $\lambda : F \to G$ is equivalent to giving for each i a morphism of sheaves $\lambda_i : F_i \to G_i$ such that $\psi_{ij} \circ \lambda_i|_{U_i \cap U_j} = \lambda_j|_{U_i \cap U_j} \circ \phi_{ij}$. The proof is straightforward.

Flasque sheaves. Let X be a topological space and let F be a sheaf on X.

Definition 1.19. *The sheaf F is flasque if for all $U, V \in \mathrm{Top}(X)$ such that $U \subseteq V$, the natural map $F(V) \to F(U)$ is surjective.*

Lemma 1.20. *Let*

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of sheaves on X. Then

(a) *If F' is flasque then the sequence*

$$0 \to F'(X) \xrightarrow{d'} F(X) \xrightarrow{d} F''(X) \to 0$$

is exact.

(b) *If F' and F are flasque then F'' is flasque.*

Proof. (a): Let $(U_i) \in \mathrm{Top}(X)$ be a family of open sets in X. We suppose that (U_i) is totally ordered by the inclusion relation (ie for two indices i, j, we have either $U_i \subseteq U_j$ or $U_j \subseteq U_i$). Let $U_{\mathrm{tot}} := \bigcup_i U_i$.

Notice that the various sequences

$$0 \to F'(U_i) \to F(U_i) \to \mathrm{Image}(d(U_i)) \to 0 \tag{10}$$

are exact. The abelian groups $F(U_i)$ naturally form an inverse system (see [Wei94, Appendix A.5] or more concretely [AM69, p. 103]). Furthermore, by the sheaf property, the natural map

$$F(U_{\mathrm{tot}}) \to \varprojlim_i F(U_i)$$

is an isomorphism. A similar statement is true for F'. Using these isomorphisms and applying $\varprojlim_i$ to the sequences (10), we obtain a sequence

$$0 \to F'(U_{\mathrm{tot}}) \to F(U_{\mathrm{tot}}) \to \varprojlim_i (\mathrm{Image}(d(U_i))) \to 0. \tag{11}$$

We assert that the sequence (11) is exact. This implies in particular that we have a natural isomorphism $\mathrm{Image}(d(U_{\mathrm{tot}})) \simeq \varprojlim_i (\mathrm{Image}(d(U_i)))$. Now it is a consequence of [AM69, Prop. 10.2, p. 104], (whose proof the reader is encouraged to read) that the inverse limit of the sequence (10) will be exact if the various arrows in the inverse system of the $F'(U_i)$ are surjective. This is true here by the assumption that F' is flasque and we have thus proven the assertion.

Fix now an element $\sigma \in F''(X)$. Consider the collection AC of open sets U in X such that in the sequence

$$0 \to F'(U) \to F(U) \to F''(U)$$

we have $\sigma|_U \in \mathrm{Image}(d(U))$. Consider the partial order on AC given by inclusion. By the assertion proven above, every totally ordered subset of AC has an upper bound. It thus follows from Zorn's lemma ([AM69, bottom of p. 3]) that there is a maximal element $U_{\max}$ in AC. If $U_{\max} = X$, we are done. Otherwise, let $x \in X \backslash U_{\max}$. Let V be an open neighbourhood of x. We choose V sufficiently small so that in the sequence

$$0 \to F'(V) \to F(V) \overset{d(V)}{\to} F''(V)$$

we have $\sigma|_V \in \mathrm{Image}(d(V))$. Now consider the sequence

$$0 \to F'(U_{\max} \cup V) \to F(U_{\max} \cup V) \overset{d(U_{\max} \cup V)}{\to} F''(U_{\max} \cup V).$$

Let $l_{U_{\max}}$ be a lifting of $\sigma|_{U_{\max}} \in F''(U_{\max})$ to $F(U_{\max})$ (ie an element such that $d(U_{\max})(l_{U_{\max}}) = \sigma|_{U_{\max}}$). Let l_V be a lifting of $\sigma|_V$. The element $(l_{U_{\max}} - l_V)_{U_{\max} \cap V}$ lies by construction in $\ker(d(U_{\max} \cap V))$. Let $e \in F'(X)$ such that

$$d'(U_{\max} \cap V)(e|_{U_{\max} \cap V}) = (l_{U_{\max}} - l_V)_{U_{\max} \cap V}$$

This exists because the mapping $F'(X) \to F(U_{\max} \cap V)$ is surjective, since F' is assumed flasque. Now replace $l_{U_{\max}}$ by $l_{U_{\max}} - d'(U_{\max})(e|_{U_{\max}})$. We now have elements $l_{U_{\max}} \in F(U_{\max})$ and $l_V \in F(V)$ lifting $\sigma|_{U_{\max}}$ and $\sigma|_V$ respectively and such that $(l_{U_{\max}} - l_V)_{U_{\max} \cap V} = 0$. By the sheaf property of F, there is thus an element $l \in F(U_{\max} \bigcup V)$ such that $l|_{U_{\max}} = l_{U_{\max}}$ and $l|_V = l_V$ and thus $d(U_{\max} \cap V)(l) = \sigma|_{U_{\max} \cup V}$.

This contradicts the maximality of $U_{\max}$ and thus we must have $X = U_{\max}$, which concludes the proof of (a).

(b): Let $U, V \in \mathrm{Top}(X)$ with $U \subseteq V$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F'(V) & \longrightarrow & F(V) & \longrightarrow & F''(V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} & & \\ 0 & \longrightarrow & F'(U) & \longrightarrow & F(U) & \longrightarrow & F''(U) & \longrightarrow & 0 \end{array}$$

Both rows in this diagram are exact by (a). Furthermore the vertical restriction arrows (1) and (2) are surjective by assumption. By the five Lemma ([Wei94, 1.3.3, p. 13]), we conclude that (3) is also surjective, which is what we want to prove. □

Lemma 1.21. *If I is an injective sheaf on X, then I is flasque.*

Proof. Let $\underline{\mathbb{Z}}$ be the sheaf on X defined by

$$\underline{\mathbb{Z}}(U) = \{\text{locally constant maps } U \to \mathbb{Z}\}$$

for $U \in \mathrm{Top}(X)$. Note that $\underline{\mathbb{Z}}(U)$ is naturally isomorphic to $\mathbb{Z}^{\mathrm{Conn}(U)}$, where $\mathrm{Conn}(U)$ is the set of connected components of U.

Let $U_0, V_0 \in \mathrm{Top}(X)$ with $U_0 \subseteq V_0$. We define $\mathcal{O}_{U_0}$ as the sheaf generated by the presheaf described by the rule

$$U \in \mathrm{Top}(X) \mapsto 0 \text{ if } U \not\subseteq U_0$$

$$U \in \mathrm{Top}(X) \mapsto \underline{\mathbb{Z}}(U) \text{ if } U \subseteq U_0$$

We have an evident exact sequence

$$0 \to \mathcal{O}_{U_0} \to \mathcal{O}_{V_0}$$

(for exactness look at the stalks). Thus, if I is an injective sheaf, we have a surjective map

$$\mathrm{Mor}(\mathcal{O}_{V_0}, I) \to \mathrm{Mor}(\mathcal{O}_{U_0}, I)$$

But there are natural identifications $\mathrm{Mor}(\mathcal{O}_{V_0}, I) \simeq I(V_0)$ and $\mathrm{Mor}(\mathcal{O}_{U_0}, I) = I(U_0)$ (to see this, first suppose that U_0 and V_0 are connected and generalise from there). We thus have a surjection

$$I(V_0) \to I(U_0)$$

which is what we wanted to prove. □

Proposition 1.22. *If F is flasque then $H^k(X, F) = 0$ for all $k > 0$.*

Proof. Suppose that F is flasque. Let $F \to I$ be an injection of F into an injective sheaf. Such an injection exists by Theorem 1.13. Consider the sequence

$$0 \to F \to I \to I/F \to 0 \tag{12}$$

where I/F is the quotient of I by F. Recall that by Lemma 1.21, the sheaf I is flasque. Now consider the long exact cohomology sequence (see Theorem 1.3) of (12). We get

$$0 \to F(X) \to I(X) \to (I/F)(X) \to H^1(X, F) \to H^1(X, I) \to \dots \tag{13}$$

Now by construction $H^1(X, I) = 0$. On the other hand by Lemma 1.20 (a), the map $I(X) \to (I/F)(X)$ is surjective. We deduce that $H^1(X, F) = 0$. Now I/F is also flasque by Lemma 1.20 (b) and Lemma 1.21. Hence $H^1(X, I/F) = 0$ as well and looking at the sequence 13 again, we see that $H^2(X, F) = 0$. Continuing in this way, we deduce that for all $k > 0$, we have $H^k(X, F) = 0$. □

Complement 1.23. (keeping the notation and assumptions of Proposition 1.22) A similar reasoning show that if $f : X \to Y$ is a continuous map of topological spaces, then $R^k f_*(F) = 0$ for all $k > 0$.

Corollary 1.24 (Leray spectral sequence)**.** *Let $f : X \to Y$, $g : Y \to Z$ be continuous map between topological spaces. Let $F^\bullet$ be a finite cochain complex of sheaves on X. Then there is an E_2 cohomological spectral sequence*

$$E_2^{pq} = R^p g_*(R^q f_*(F^\bullet)) \Rightarrow R^{p+q}(g \circ f)_*(F^\bullet)$$

which is functorial in $F^\bullet$.

Proof. Let $I^\bullet$ be an injective resolution of $F^\bullet$. Since the I^k are flasque by Lemma 1.21, the sheaves $f_*(I^k)$ are also flasque. Hence by Exercise 1.5, we have

$$R^{p+q} g_*(f_*(I^\bullet)) = \mathcal{H}^{p+q}((g \circ f)_*(I^\bullet)) = R^{p+q}(g \circ f)_*(F^\bullet)$$

and Theorem 1.4 gives us a spectral sequence

$$E_2^{pq} = R^p g_*(\mathcal{H}^p(f_*(I^\bullet))) = R^p g_*(R^q f_*(F^\bullet)) \to R^{p+q} g_*(f_*(I^\bullet)) = R^{p+q}(g \circ f)_*(F^\bullet).$$

□

1.4 Exercises

Exercise 1.1 (Yoneda's lemma). *Let $\mathcal{C}$ be a category. Let $\mathbf{Sets}^{\mathcal{C}^{\rm opp}}$ be the category of set-valued contravariant functors from $\mathcal{C}$ to $\mathbf{Sets}$. Prove that $\mathrm{Mor}_{\mathcal{C}}(\bullet, C)$ defines a contravariant functor $h_C : \mathcal{C} \to \mathbf{Sets}$ for each object C of $\mathcal{C}$. Prove that $h_\bullet$ defines a fully faithful functor $\mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{\rm opp}}$.*

Exercise 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Let $L : \mathcal{B} \to \mathcal{A}$ (resp. $R : \mathcal{A} \to \mathcal{B}$) be additive functors. Suppose that L is left adjoint to R (see* [Wei94, Appendix A.6]*). Then L is right-exact and R is left-exact.*

Exercise 1.3. *Prove Lemma 1.1.*

Exercise 1.4. *Prove Lemma 1.2.*

Exercise 1.5. *Show that Theorem 1.4 implies Theorem 1.3. Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor to another abelian category. We say that an object A of $\mathcal{A}$ is F-**acyclic** if $R^k F(A) = 0$ for all $k > 0$. Show that if A is an object of $\mathcal{A}$ and $C^\bullet$ is a resolution of A, such that C^k is F-acyclic for all $k \in \mathbb{Z}$, then there is a natural isomorphism $R^k F(A) \simeq \mathcal{H}^k(C^\bullet)$ for all $k \in \mathbb{Z}$.*

Exercise 1.6. *Prove the assertions after Definition 1.7.*

Exercise 1.7. *Let $\phi : F \to G$ be a morphism of sheaves on a topological space X. In the text, we defined the kernel $\ker(\phi)$ of ϕ as the presheaf*

$$U \in \mathrm{Top}(X) \;\mapsto\; \ker(\phi(U))$$

Prove that $\ker(\phi)$ is a sheaf.

Exercise 1.8. *Prove Proposition 1.10.*

Exercise 1.9. *Let $f : X \to Y$ be a topological space. Prove from the definition that $f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ is a left exact functor. Prove that f^{-1} is an exact functor.*

Exercise 1.10. *Show that an abelian group G is injective in the category $\mathbf{Ab}$ if G is divisible (ie for all $n \in \mathbb{Z}\backslash\{0\}$, the 'multiplication by n' map $G \to G$ is surjective). Show that the category $\mathbf{Ab}$ has enough injectives.*

Exercise 1.11. *Let $C^\bullet$ be a cochain complex of sheaves on a topological space X. Suppose that for all $U \in \mathrm{Top}(X)$, the complex $C^\bullet(U)$ is exact. Prove that $C^\bullet$ is exact.*

2 Schemes. Quasi-coherent sheaves.

A *ringed space* is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. The ringed space $(X, \mathcal{O}_X)$ is said to be *locally ringed* if the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$. In that case we will often write $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ for the maximal ideal and $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ for the residue field of $\mathcal{O}_{X,x}$.

A *morphism of ringed spaces* $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed, we say that $(f, f^\#)$ is *local* or that it is *a morphism of locally ringed spaces* if for all $x \in X$, the induced map of stalks $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is a local morphism of rings.

Recall that a morphism of local rings $\phi : R \to T$ is said to be local if $\phi(\mathfrak{m}_R) \subseteq (\mathfrak{m}_T)$. Here $\mathfrak{m}_T$ (resp. $\mathfrak{m}_R$) is maximal ideal of T (resp. R).

If $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\#) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces, the composition $(h, h^\#) = (g, g^\#) \circ (f, f^\#) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is defined in the following way. We let $h := g \circ f$ (in the sense of composition of maps). The morphism of sheaves $h^\# : \mathcal{O}_Z \to h_*(\mathcal{O}_X)$ is defined as the unique morphism $h^\#$ making the following diagram commutative:

$$\begin{array}{ccc} g_*(\mathcal{O}_Y) & \xrightarrow{g_*(f^\#)} & g_*(f_*(\mathcal{O}_X)) \\ {\scriptstyle g^\#}\uparrow & & \uparrow{\scriptstyle \simeq} \\ \mathcal{O}_Z & \xrightarrow{h^\#} & (g \circ f)_*(\mathcal{O}_X) = h_*(\mathcal{O}_X) \end{array}$$

2.1 Affine schemes

Let R be a ring. We define $\operatorname{Spec}(R)$ as the set of prime ideals of R. If $\mathfrak{a} \subseteq R$ is an ideal, we define

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(R) \,|\, \mathfrak{p} \supseteq \mathfrak{a}\}$$

Lemma 2.1. *The symbol $V(\bullet)$ has the following properties:*

- $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b})$;
- $\bigcap_{i \in I} V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$;
- $V(R) = \emptyset$; $V((0)) = \operatorname{Spec}(R)$.

Proof. This is Exercise 2.2. □

An immediate consequence of Lemma 2.1 is that the sets $V(\mathfrak{a})$ ($\mathfrak{a}$ an ideal of R) form the closed sets of a topology on $\operatorname{Spec}(R)$. This topology is called the *Zariski topology*. The closed points in $\operatorname{Spec}(R)$ are precisely the maximal ideals of R.

Lemma 2.2. *Let $f \in R$. The set*

$$D_f(R) = D_f = \{\mathfrak{p} \in \operatorname{Spec}(R) \,|\, f \notin \mathfrak{p}\}$$

is open in $\operatorname{Spec}(R)$. The open sets of $\operatorname{Spec}(R)$ of the form D_f form a basis for the Zariski topology of $\operatorname{Spec}(R)$. Furthermore, the topology of $\operatorname{Spec}(R)$ is quasi-compact.

The open sets of the form D_f are often called *basic open sets* (in $\mathrm{Spec}(R)$). Recall that a set B of open sets of a topological space X is said to be a *basis* for the topology of X if every open set of X can be written as a union of open sets in B. A topological space X is called *quasi-compact* if: for every family $(U_{i\in I})$ of open sets in X such that $\bigcup_{i\in I} U_i = X$ there exists a finite subset $I_0 \subseteq I$ such that $\bigcup_{i\in I_0} U_i = X$.

Proof. We shall prove that D_f is open. To see this, just notice that the complement of D_f in $\mathrm{Spec}(R)$ is precisely $V((f))$, where (f) is the ideal generated by f.

We now prove that the open sets of $\mathrm{Spec}(R)$ of the form D_f form a basis for the Zariski topology of $\mathrm{Spec}(R)$. Let $\mathfrak{a}$ be an ideal. We have to show that the set

$$\mathrm{Spec}(R)\backslash V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \mathfrak{p} \not\supseteq \mathfrak{a}\}$$

is equal to $\bigcup_{i\in I} D_{r(i)}$ for some index set I and some function $r : I \to R$. Let $r : I \to \mathfrak{a}$ be an enumeration of a set of generators of $\mathfrak{a}$. In view of Lemma 2.1, we have the required equality.

Finally, we show that $\mathrm{Spec}(R)$ is quasi-compact. In view of the fact that the open sets of $\mathrm{Spec}(R)$ of the form D_f form a basis for the Zariski topology of $\mathrm{Spec}(R)$, we only need to show that if

$$\mathrm{Spec}(R) = \bigcup_{i\in I} D_{r(i)} \tag{14}$$

where $r : I \to R$ is a some function, then there is a finite subset $I_0 \subseteq I$ such that $\mathrm{Spec}(R) = \bigcup_{i\in I_0} D_{r(i)}$. Now notice that by Lemma 2.1 and the proof of the first statement of the present lemma, the equality (14) is equivalent to the equality

$$\bigcap_{i\in I} V((r(i))) = V((r(I))) = \emptyset \tag{15}$$

where we have used the short-hand $(r(I))$ for the ideal generated by all the $r(i)$. Now the equality $V((r(I))) = \emptyset$ says that no prime ideal contains $(r(I))$. This is only possible if $(r(I)) = R$, for otherwise $(r(I))$ would be contained in at least one maximal ideal and maximal ideals are prime (see [AM69, I, Cor. 1.4]). Now choose a finite subset $I_0 \subseteq I$ and a map $c : I_0 \to R$ such that $1 = \sum_{i\in I_0} c(i) \cdot r(i)$. We then have $\sum_{i\in I_0}(r(i)) = R$ and thus $\bigcap_{i\in I_0} V((r(i))) = \emptyset$, which is what we want. □

Lemma 2.3. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals in R. Then $V(\mathfrak{a}) = V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.*

Here $\sqrt{\mathfrak{a}}$ is the *nilradical* of $\mathfrak{a}$ (see [AM69, p. 5]).

Proof. Exercise 2.3. □

In particular, there is a one to one correspondence between radical ideals in R and closed subsets of $\mathrm{Spec}(R)$. If $\mathfrak{a}, \mathfrak{b}$ are radical ideals then $\mathfrak{a} \subseteq \mathfrak{b}$ if and only if $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$. Recall that an ideal $\mathfrak{a}$ is called *radical* if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

Remark 2.4. Let R be a commutative ring and let $\mathfrak{a}, \mathfrak{b}$ be two ideals in R. Then we have

$$(\mathfrak{a} \cap \mathfrak{b}) \cdot (\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathfrak{a} \cdot \mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$$

and thus $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}}$. In particular, we have

$$V(\mathfrak{a} \cdot \mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}).$$

Note that if $\mathfrak{a}$ and $\mathfrak{b}$ are radical ideals then $\mathfrak{a} \cap \mathfrak{b}$ is also a radical ideal, whereas $\mathfrak{a} \cdot \mathfrak{b}$ might not be.

We wish to make $\mathrm{Spec}(R)$ into a locally ringed space. We define a sheaf of rings on $\mathrm{Spec}(R)$ as follows. For $U \in \mathrm{Top}(\mathrm{Spec}(R))$, let

$$\begin{aligned}\mathcal{O}_{\mathrm{Spec}(R)}(U) := \{s : U \to \coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} \mid & \text{ for all } \mathfrak{p} \in U \text{ we have } s(\mathfrak{p}) \in R_{\mathfrak{p}} \\ & \text{and for all } \mathfrak{p} \in U \text{ there is } a, r \in R \text{ and } V \in \mathrm{Top}(U) \\ & \text{such that } D_r(R) \supseteq V \text{ and } s(\mathfrak{q}) = \frac{a}{r} \text{ for all } \mathfrak{q} \in V\} \qquad (16)\end{aligned}$$

This formula clearly defines a sheaf on rings on $\mathrm{Spec}(R)$.

Proposition 2.5. (a) *For all $r \in R$, we have a canonical isomorphism $\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R)) \simeq R_r$.*

(b) *If $t \in R$ and $t \in (r)$ then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spec}(R)}(D_r) & \longrightarrow & \mathcal{O}_{\mathrm{Spec}(R)}(D_t) \\ \downarrow \simeq & & \downarrow \simeq \\ R_r & \longrightarrow & R_t \end{array}$$

where the vertical isomorphisms come from (a).

(c) *There is a natural isomorphism $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \simeq R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. This isomorphism fits in a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} & \xrightarrow{\simeq} & R_{\mathfrak{p}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) & \xrightarrow{\simeq} & R \end{array}$$

Here the vertical morphisms are the natural ones and the lower horizontal one comes from (a).

Proof. (a): there is a map $Q : R_r \to \mathcal{O}_{\mathrm{Spec}(R)}(D_r(R))$ given by the formula

$$Q(\frac{v}{r}) = \text{map } s : \mathrm{D}_r \to \coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} \text{ such that } s(\mathfrak{p}) \text{ is the image of } \tfrac{v}{r} \text{ in } R_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in D_r.$$

This is the sought canonical map. We wish to show that this map is an isomorphism.

The map Q is injective. The kernel of Q consists of elements $v/r \in R_r$ such that the image of v/r in $R_{\mathfrak{p}}$ vanishes for all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $r \notin \mathfrak{p}$. Now suppose that $v/r \in \ker(Q)$ and that $v/r \neq 0$. Let

$$\mathrm{Ann}(v/r) := \{z \in R_r \mid z \cdot \frac{v}{r} = 0\}$$

This is an ideal of R_r, called the *annihilator* of v/r. Since $v/r \neq 0$, the ideal $\mathrm{Ann}(v/r)$ is not equal to R_r. Let $\mathfrak{m}$ be a maximal ideal containing $\mathrm{Ann}(v/r)$ (see [AM69, Cor. 1.4, p . 4] for this). Then the image of v/r in $R_{\mathfrak{m}}$ does not vanish by construction. Thus v/r must vanish.

The map Q is surjective. Suppose given $s \in \mathcal{O}_{\mathrm{Spec}(R)}(D_r)$. We now that in the neighbourhood of every point of D_r, the map s is represented by a fraction (in the sense of (16)). Since the sets of the form D_l form a basis for the topology of $\mathrm{Spec}(R)$ and since D_r is quasi-compact (see Remark 2.6 below)), there are $r_1, \dots r_n \in R$ such that $D_{r_i} \subseteq D_r$ for all r_i, such that $\bigcup_i D_{r_i} = D_r$ there are $a_1, \dots a_n \in R$ such that s is represented on D_{r_i} by a_i/r_i. Now $D_{r_i} \cap D_{r_j} = D_{r_i r_j}$ and thus using the fact that Q is injective, we see that we have

$a_i/r_i = a_j/r_j$ in $R_{r_i r_j}$. By the definition of localisation, this means that there is $l \geqslant 0$, which may be taken to be independent of i, j, such that

$$(r_i r_j)^l r_j a_i = (r_i r_j)^l r_i a_j$$

Now using the assumption and Lemma 2.3, we see that there are $b_1, \dots, b_n \in R$ and $e \geqslant 1$ such that

$$r^e = \sum_i b_i r_i^{l+1}.$$

Let $v := \sum_i b_i r_i^l a_i$. We compute

$$v r_j^{l+1} = \sum_i b_i r_i^l r_j^{l+1} a_i = \sum_i b_i r_i^{l+1} r_j^l a_j = r^e r_j^l a_j$$

so that

$$\frac{v}{r^e} = \frac{a_j}{r_j}$$

in R_{r_j}. In other words, v/r^e is an element of D_r whose image in $\mathcal{O}_{\mathrm{Spec}(R)}(D_r)$ is s.

(b): unwind the definitions.

(c): in view of Lemma 2.2, we have a natural isomorphism

$$\varinjlim_{r \in R; r \notin \mathfrak{p}} \mathcal{O}_{\mathrm{Spec}(R)}(D_r)$$

By (a) and (b), this ring is naturally isomorphic to $\varinjlim_{r \in R; r \notin \mathfrak{p}} R_r$, which can be identified with $R_{\mathfrak{p}}$. See Exercise 2.4. $\square$

Suppose now given a morphism of rings $\phi : R \to T$. We obtain a continuous map $\mathrm{Spec}(\phi) : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ by the formula

$$\mathrm{Spec}(\phi)(\mathfrak{p}) := \phi^{-1}(\mathfrak{p})$$

Remark 2.6. Notice for $r \in R$, we have $D_r(R) = \mathrm{Spec}(R \to R_r)(\mathrm{Spec}(R_r))$. This shows that $D_r(R)$ is also quasi-compact.

Furthermore, we define a morphism of sheaves of rings

$$\phi^{\#} : \mathcal{O}_{\mathrm{Spec}(R)} \to \mathrm{Spec}(\phi)_*(\mathcal{O}_{\mathrm{Spec}(T)})$$

as follows. By the sheaf property, it is sufficient to provide for all $r \in R$ maps of rings

$$\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R)) \to \mathrm{Spec}(\phi)_*(\mathcal{O}_{\mathrm{Spec}(T)})(D_r(R))$$

which are compatible with the inclusion maps $D_t \to D_r$ when $t \in (r)$. Now we have $\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R)) \simeq R_r$ and

$$\mathrm{Spec}(\phi)_*(\mathcal{O}_{\mathrm{Spec}(T)})(D_r(R)) \simeq T_{\phi(r)}$$

by Proposition 2.5(a) and the definition of the direct image sheaf. Furthermore, there is a natural map of rings $R_r \to T_{\phi(r)}$, which is induced by ϕ. This map is obviously compatible with inclusion maps $D_t \to D_r$ when $t \in (r)$. Using Proposition 2.5(b), we conclude that we have indeed obtained for all $r \in R$ maps of rings

$$\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R)) \to \mathrm{Spec}(\phi)_*(\mathcal{O}_{\mathrm{Spec}(T)})(D_r(R))$$

which are compatible with the inclusion maps $D_t \to D_r$ when $t \in (r)$.

All in all we have associated with any ring R a locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ and we have associated with any morphism $\phi : R \to T$ of rings a morphism of ringed spaces $(\mathrm{Spec}(\phi), \phi^{\#})$, which can easily be shown to be local using Proposition 2.5(c). We have in fact defined a *contravariant* functor from the category of rings to the category of locally rings spaces. We skip the details (which are not difficult to verify) that the process that we have described is indeed functorial.

Lemma 2.7. *The above functor is fully faithful.*

Proof. We start with a morphism of locally ringed spaces

$$(f, f^{\#}) : (\mathrm{Spec}(T), \mathcal{O}_{\mathrm{Spec}(T)}) \to (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

We are thus given a morphism of sheaves of rings

$$\mathcal{O}_{\mathrm{Spec}(R)} \to f_*(\mathcal{O}_{\mathrm{Spec}(T)})$$

and thus a morphism of rings

$$\phi : R \simeq \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \to f_*(\mathcal{O}_{\mathrm{Spec}(T)})(\mathrm{Spec}(R)) \simeq T$$

we shall be done if we can show that $(f, f^{\#}) = (\mathrm{Spec}(\phi), \phi^{\#})$. We shall first show that $f = \mathrm{Spec}(\phi)$. We need to show that $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(T)$. Now we know that the morphism of rings

$$f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\mathrm{Spec}(R), f(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}(T), \mathfrak{p}}$$

is local (because $(f, f^{\#})$ is a morphism of *locally* ringed spaces). In view of Proposition 2.5(c), this morphism fits in a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\phi} & T \\ \downarrow{\scriptstyle l_{f(\mathfrak{p})}} & & \downarrow{\scriptstyle l_{\mathfrak{p}}} \\ \mathcal{O}_{\mathrm{Spec}(R), f(\mathfrak{p})} & \xrightarrow{f_{\mathfrak{p}}^{\#}} & \mathcal{O}_{\mathrm{Spec}(T), \mathfrak{p}} \end{array}$$

where the vertical maps are the localisation maps. We compute

$$\phi^{-1}(\mathfrak{p}) = \phi^{-1}(l_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(T),\mathfrak{p}}})) = l^{-1}_{f(\mathfrak{p}}(f_{\mathfrak{p}}^{\#,-1}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(T),\mathfrak{p}}})) = l^{-1}_{f(\mathfrak{p})}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(R),f(\mathfrak{p})}}) = f(\mathfrak{p}).$$

Here we have used the fact that $f_{\mathfrak{p}}^{\#}$ is local in the third equality. The diagram also shows that $f_{\mathfrak{p}}^{\#} = \phi_{\mathfrak{p}}$. Hence, we see that the morphisms of sheaves $\phi^{\#}$ and $f^{\#}$ coincide on the stalks. This shows that there are equal. □

A locally ringed space isomorphic to a space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called *affine*.

We shall write Aff for the category of affine schemes and CRings for the category of unital commutative rings.

2.2 Schemes

Definition 2.8. *A scheme is a locally ringed space X such that every point x in X has an open neighbourhood U, which is isomorphic to an affine scheme as a locally ringed space. A morphism of schemes is a morphism of locally ringed spaces.*

We shall write **Schemes** for the category of schemes.

Definition 2.9. *A scheme X is locally noetherian it is has an open covering $(U_{i\in I} \in \mathrm{Top}(X))$ such that each U_i is isomorphic to an affine scheme $(\mathrm{Spec}(R_i), \mathcal{O}_{\mathrm{Spec}(R_i)})$, where R_i is a noetherian ring.*

Recall that a ring is *noetherian*, if every ideal of R is finitely generated as an R-module. This is equivalent to the following properties:

- if M is a finitely generated R-module and N is a submodule of M, then N is also finitely generated;
- if $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ is an ascending family of ideals of R, then there is an index i such that

$$I_i = I_{i+1} = I_{i+2} = \dots$$

These equivalences are the object of Exercise 2.5. It is also shown there that the localisation of a noetherian ring is noetherian.

Proposition 2.10. *A scheme X is locally noetherian if and only if for any open subset U of X, which is isomorphic to an affine scheme $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ as a locally ringed space, the ring R is noetherian.*

Proof. By simple logical reductions using Exercise 2.5 and Lemma 2.2, the statement of the theorem can be shown to be equivalent to the following statement of commutative algebra. Let R be a ring. Let $f_1, \dots, f_n \in R$ be such that the ideal $(f_1, \dots f_n)$ generated by the f_i is R. If R_{f_i} is noetherian for all i, then R is noetherian. This is what we shall prove.

So let $J \subseteq R$ be an ideal. Let $\phi_i : R \to R_{f_i}$ be the natural maps. We shall first prove the equality

$$J = \bigcap_{i\in\{1,\dots,n\}} \phi_i^{-1}(\phi_i(J)R_{f_i}) \tag{17}$$

Here $\phi_i(J)R_{f_i}$ is the ideal in R_{f_i} generated by $\phi_i(J)$. We clearly have

$$J \subseteq \bigcap_{i\in\{1,\dots,n\}} \phi_i^{-1}(\phi_i(J)R_{f_i}).$$

For the reverse inclusion, let $b \in \bigcap_{i\in\{1,\dots,n\}} \phi_i^{-1}(\phi_i(J)R_{f_i})$. For each index i, let $a_i \in R$ and $m_i \geqslant 0$ be such that $\phi_i(b) = a_i/f_i^{m_i}$, where $a_i \in J$. My may assume wrog that all m_i are equal to one $m \in \mathbb{N}$. There is then one $k \in \mathbb{N}$ such that

$$f_i^k(f_i^m b - a_i) = 0$$

for all indices i. Hence $f_i^{k+m} b \in J$ for all i. Now from the assumption that $(f_1, \dots f_n) = R$ and Lemma 2.3, we see that there are elements $c_i \in R$ such that

$$\sum_i c_i f_i^{k+m} = 1$$

Thus $b \in J$, establishing the equality (17). Now consider an ascending sequence of ideals

$$J_0 \subseteq J_1 \subseteq \dots$$

of ideals of R. For each index i, we have

$$\phi_i(J_0)R_{f_i} \subseteq \phi_i(J_1)R_{f_i} \subseteq \dots \tag{18}$$

and since the R_{f_i} are assumed noetherian, the sequence (18) becomes stationary at an index k_0, which may be chosen independently of i. Using (17), we conclude that the sequence (2.2) also becomes stationary at the index k_0. □

A scheme X is *noetherian* if it is quasi-compact as a topological space and locally noetherian.

A topological space T is called *noetherian* if for any descending sequence

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$$

of closed subsets of T, there is an index i such that

$$C_i = C_{i+1} = C_{i+2} = \dots$$

Lemma 2.11. *A noetherian topological space is quasi-compact.*

Proof. Exercise 2.17. □

We record the following lemma, which is a consequence of the definitions:

Lemma 2.12. *A noetherian scheme is noetherian as a topological space.*

A scheme X is *reduced* if for all $U \in \mathrm{Top}(X)$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. A scheme X is *integral*, if for all $U \in \mathrm{Top}(X)$, the ring $\mathcal{O}_X(U)$ is a domain (also called an integral ring).

Properties of morphisms of schemes.

Let X be a scheme. An *open affine covering* $(U_{i\in I})$ of X is a family of open subsets U_i of X such that

- $\bigcup_i U_i = X$;
- if U_i is endowed with the structure of locally ringed space coming from X, then U_i is an affine scheme.

Let $(f, f^\#) : X \to Y$ be a morphism of schemes.

- $(f, f^\#)$ is *quasi-compact* if there is an open affine covering (V_i) of Y such that $f^{-1}(V_i)$ is quasi-compact for all i.
- $(f, f^\#)$ is *locally of finite type* if f there is a an open affine covering (V_i) of Y and for each i an open affine covering (U_{ij}) of $f^{-1}(V_i)$ such that $\mathcal{O}_X(U_{ij})$ is a finitely generated $\mathcal{O}_Y(V_i)$-algebra via the morphism $(f, f^\#)$.
- $(f, f^\#)$ is *of finite type* of it is quasi-compact and locally of finite type.
- $(f, f^\#)$ is *a closed immersion* if the following conditions are satisfied: the image of f is closed, f is a homeomorphism of X onto $f(X)$ and the morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. We then say that X is a *closed subscheme* of Y via $(f, f^\#)$ or simply that $f(X)$ is a closed subscheme of Y.
- $(f, f^\#)$ is *an open immersion* if $f(X)$ is open, f is a homeomorphism onto its image and the mapping of stalks $f^\#_y : \mathcal{O}_y \to (f_*\mathcal{O}_X)_y$ is an isomorphism for all $y \in f(X)$. We then say that X is an *open subscheme* of Y via $(f, f^\#)$ or simply that $f(X)$ is an open subscheme of X.

- $(f, f^{\#})$ is *affine* if there is an open affine covering (V_i) of Y such that $f^{-1}(V_i)$ is affine for all i.

The notion of affine morphism, morphism locally of finite type and quasi-compact morphism are independent of the affine covering appearing in their definitions: see Exercise 2.6.

We shall often use the short-hand

'$f : X \to Y$ is a morphism of schemes'

for

'$(f, f^{\#}) : X \to Y$ is a morphism of schemes'.

The next Proposition explains how to glue schemes.

Suppose given (U_i) a family of schemes and for each pair of indices ij an open subscheme $U_{ij} \to U_i$

Suppose given isomorphisms $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ for all indices i, j, satisfying the properties (1), (2), (3) below.

(1) $U_{ii} = U_i$;

(2) $\phi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{jk}$;

(3) $\phi_{ik}|_{U_{ij} \cap U_{ik}} = \phi_{jk} \circ \phi_{ij}|_{U_{ij} \cap U_{ik}}$ as morphisms $U_{ij} \cap U_{ik} \to U_k$.

for all indices i, j, k.

If the (U_i) are open subschemes of a given scheme X, then we may define

$$U_{ij} := U_i \cap U_j$$

and define the morphisms ϕ_{ij} in the obvious way. These ϕ_{ij} satisfy (1), (2), (3) by construction. The following proposition is a converse.

Proposition 2.13. *There is up to unique isomorphism a scheme X with the following properties. There are open immersions $\psi_i : U_i \to X$ such that $\bigcup_i \psi_i(U_i) = X$ and such that $\psi_j \circ \phi_{ij} = \psi_i|_{U_{ij}}$.*

Proof. See Exercise 2.7. □

The following lemma records how to glue morphisms.

Lemma 2.14. *Let X and Y be schemes. Let (U_i) be a covering of X by open subschemes. For all indices i, j, let $(u_{ij}, u_{ij}^{\#}) : U_{ij} := U_i \cap U_j \to U_i$ be the natural open immersion. To give a morphism $(f, f^{\#}) : X \to Y$ is equivalent to giving morphisms $(f_i, f_i^{\#}) : U_i \to Y$ for all i with the property that $(f_i, f_i^{\#}) \circ (u_{ij}, u_{ij}^{\#}) = (f_j, f_j^{\#}) \circ (u_{ji}, u_{ji}^{\#})$ for all i, j.*

Proof. If a morphism f is given, then the morphisms $(f_i, f_i^{\#})$ obtained by restricting $(f, f^{\#})$ to U_i will have the advertised property. On the other hand, suppose given a family of $(f_i, f_i^{\#}) : U_i \to Y$ with the advertised property. We define in the obvious manner a map $f : X \to Y$ restricting to the various f_i. It remains to define a morphism of sheaves of rings

$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$

Notice that it is equivalent to give a morphism of sheaves of rings $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ (see Proposition 1.11). Now we have morphisms of sheaves of rings

$$f^{-1}(\mathcal{O}_Y)|_{U_i} = f_i^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$$

coming from the morphisms $(f_i, f_i^{\#})$. The fact that for all i, j we have

$$(f_i, f_i^{\#}) \circ (u_{ij}, u_{ij}^{\#}) = (f_j, f_j^{\#}) \circ (u_{ji}, u_{ji}^{\#})$$

implies that the morphisms $f^{-1}(\mathcal{O}_Y)|_{U_i} \to \mathcal{O}_X$ glue to give a global morphism of sheaves $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ in the sense of Complement 1.18 (we encourage the reader to write out the relevant commutative diagrams). □

Lemma 2.15. *Let X, Y be schemes. The functor $\mathrm{Top}(X) \to \mathbf{Sets}$ such*

$$U \in \mathrm{Top}(X) \mapsto \mathrm{Mor}(U, Y)$$

is a sheaf.

Proof. This is Exercise 2.18. □

Fibre products of schemes.

Let $\mathcal{C}$ be a category. Let (C_i) $(i = 1, \ldots, n)$ be a finite family of objects in $\mathcal{C}$. Recall that the *product* (resp. *coproduct*)

$$C_1 \times \cdots \times C_n = \prod_i C_i$$

(resp.

$$C_1 \coprod C_2 \coprod \cdots \coprod C_n = \coprod_i C_i)$$

of the C_i (it it exists) is an object P of $\mathcal{C}$ together with arrows $\pi_i : P \to C_i$ (resp. $\pi_i : C_i \to P$), characterised by the following property. If P' is another object together with arrows $\pi_i' : P' \to C_i$ (resp. $\pi_i' : C_i \to P'$), then there is a unique arrow $u : P' \to P$ (resp. $u : P \to P'$) such that $\pi_i \circ u = \pi_i'$ (resp. $u \circ \pi_i = \pi_i'$) for all i. By its very definition, the product P is unique up to unique isomorphism. Notice also that the coproduct of the C_i is just the product of the C_i viewed as objects of the category $\mathcal{C}^{\mathrm{opp}}$ opposite to $\mathcal{C}$. See [Wei94, Appendix A.5, p. 428] for all this.

If C is an object of $\mathcal{C}$, we shall write $\mathcal{C}/C$ for the following category. The objects of $\mathcal{C}/C$ are morphisms $D \to C$ in $\mathcal{C}$. A morphism from $\phi : D \to C$ to $\lambda : E \to C$ is a morphism $\mu : D \to E$ such that $\lambda \circ \mu = \phi$.

The morphism μ, viewed as a morphism in $\mathcal{C}$, is often called a *C-morphism*. The category $\mathcal{C}/C$ is called the *category of C-objects* (associated with $\mathcal{C}$ and C).

One often writes $D \times_C E$ for the product of $D \to C$ and $E \to C$ in $\mathcal{C}/C$ (if it exists). It is sometimes called the *fibre product* of D and E over C.

Proposition 2.16. *Let S be a scheme. Finite products exist in $\mathbf{Schemes}/S$.*

Proof. Sketch. It is sufficient to prove the result for $n = 2$. So suppose that we are given schemes X_1, X_2, S and morphisms $X_1 \to S$, $X_2 \to S$. Suppose first that X_1, X_2 and S are all affine schemes. Let $\mathrm{Spec}(B_1) =$

X_1, $\mathrm{Spec}(B_2) = X_2$, $S = \mathrm{Spec}(A)$.The tensor product of A-algebras $B_1 \otimes_A B_2$ (see [AM69, p. 30]) is by definition a coproduct in the category of A-algebras. In view of Lemma 2.7, we see that

$$X := (\mathrm{Spec}(B_1 \otimes_A B_1), \mathcal{O}_{\mathrm{Spec}(B_1 \otimes_A B_1)})$$

together with the scheme morphisms coming from the ring morphisms

$$B_1 \overset{b_1 \mapsto b_1 \otimes 1}{\rightarrow} B_1 \otimes_A B_2$$

and

$$B_2 \overset{b_1 \mapsto 1 \otimes b_2}{\rightarrow} B_1 \otimes_A B_2$$

is a product in the full subcategory of **Schemes**$/S$ consisting of affine schemes.

We shall show that X is in fact a product in the category **Schemes**$/S$. Suppose given another S-scheme $X' \to S$ and S-morphisms $\pi_1' : X' \to X_1$ and $\pi_2' : X' \to X_2$. Let (U_i) be an open affine covering of X'. By restriction U_i also comes with two morphisms $U_i \to X_1$ and $U_i \to X_2$ and since X is a product in the category of S-schemes, which are affine, we get a unique morphism $\rho_i : U_i \to X$, such that ρ_i composed with $X \to X_1$ (resp. $X \to X_2$) is $U_i \to X_1$ ($U_i \to X_2$). On $U_i \cap U_j$, the morphisms ρ_i and ρ_j coincide by unicity and thus the morphisms glue to a S-morphism $\rho : X' \to X$ (use Lemma 2.14). This shows that X is a product in the full category **Schemes**$/S$.

In general the fibre product $X_1 \times_S X_2$ is constructed as follows. Take affine coverings (U_i) of X_1 (resp. (V_j) of X_2). The schemes $U_i \times_S V_j$ come with natural glueing data satisfying the properties stated before Proposition 2.13 and we may glue them together using Proposition 2.13. See [Har77, Th. II.3.3, p. 87] for the details. □

Suppose that $X \to T$ and $S \to T$ are scheme morphisms. The scheme $X \times_T S$ together with the natural morphism $X \times_T S \to S$ is often called the *base-change of* $X \to T$ to S. One also writes X_S instead of $X \times_T S$.

Lemma 2.17. *Let $(X, \mathcal{O}_X)$ be a scheme and let R be a ring. Show that there is a canonical morphism of schemes*

$$(g, g^\#) : (X, \mathcal{O}_X) \to (\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)), \mathcal{O}_{\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))})$$

and that every morphism from $(X, \mathcal{O}_X)$ to an affine scheme factors uniquely through $(g, g^\#)$.

Proof. See Exercise 2.14. □

2.3 Sheaves of modules

Let X be a ringed space. An *$\mathcal{O}_X$-module* or *sheaf in $\mathcal{O}_X$-modules* is an abelian sheaf F, together with a $\mathcal{O}_X(U)$-module structure on $F(U)$ for every open set $U \subseteq X$, subject to obvious compatibility properties with respect to inclusions $U \to V$ of open sets in X. A *morphism of $\mathcal{O}_X$-modules* $F \to G$ is a morphism of abelian sheaves compatible with the $\mathcal{O}_X$-module structure in an obvious sense. The $\mathcal{O}_X$-modules form an additive category $\mathrm{Mod}_{\mathcal{O}_X}(X)$, which is abelian. The proof is similar to the proof that $\mathbf{Ab}(X)$ is abelian and will be skipped. We leave it as an exercise to prove the following fact: a sequence in $\mathrm{Mod}_{\mathcal{O}_X}(X)$ is exact if and only if the corresponding sequence in $\mathbf{Ab}(X)$ (obtained by forgetting the module structures) is exact.

Let F and G be $\mathcal{O}_X$-modules on X. The tensor product $F \otimes_{\mathcal{O}_X} G$ is the sheaf generated by the presheaf on X given by the formula

$$U \mapsto F(U) \otimes_{\mathcal{O}_X(U)} G(U)$$

This sheaf has a unique structure of $\mathcal{O}_X$-module, such that the map

$$F(U) \otimes_{\mathcal{O}_X(U)} G(U) \to (F \otimes_{\mathcal{O}_X} G)(U)$$

is a map of $\mathcal{O}_X(U)$-modules for every $U \in \mathrm{Top}(X)$ (details left to the reader).

Suppose $f : X' \to X''$ is a continuous map of topological spaces and that X'' is ringed by the sheaf of rings $\mathcal{O}_{X''}$. Let F be a sheaf in $\mathcal{O}_{X''}$-modules on X''. The abelian sheaf $f^{-1}(\mathcal{O}_{X''})$ is then endowed with a canonical structure of sheaf of rings, as can be seen by looking at its definition. Furthermore, the abelian sheaf $f^{-1}(F)$ inherits an obvious $f^{-1}(\mathcal{O}_{X''})$-module structure from the $\mathcal{O}_{X''}$-module structure of F on X'' (we invite the reader to go through the details of all this).

Let $f : Z \to X$ be a morphism of ringed spaces. Let F be a $\mathcal{O}_X$-module. We define

$$f^*(F) := f^{-1}(F) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Z.$$

Here $\mathcal{O}_Z$ is viewed as a $f^{-1}(\mathcal{O}_X)$-module through the canonical map of sheaves of rings $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Z$. For each $U \in \mathrm{Top}(Z)$, the group

$$f^{-1}(F)(U) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Z(U)$$

has a $\mathcal{O}_Z(U)$-module structure, which comes from the action of $\mathcal{O}_Z(U)$ on the second factor. There is a unique structure of $\mathcal{O}_Z$-module on $f^*(F)$ such that for all $U \in \mathrm{Top}(Z)$, the map

$$f^{-1}(F)(U) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Z(U) \to (f^{-1}(F) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Z)(U) = f^*(U)$$

is a map of $\mathcal{O}_Z(U)$-modules. The proof is straightforward.

Let now F be a $\mathcal{O}_Z$-module. The abelian sheaf $f_*(F)$ is naturally a sheaf in $f_*(\mathcal{O}_Z)$-modules. Via the morphism of sheaves of rings $\mathcal{O}_Z \to f_*(\mathcal{O}_Z)$, we may thus view $f_*(F)$ as a $\mathcal{O}_X$-module.

Lemma 2.18. *The functor* $f^* : \mathrm{Mod}_{\mathcal{O}_X}(X) \to \mathrm{Mod}_{\mathcal{O}_Z}(Z)$ *is left-adjoint to the functor*

$$f_* : \mathrm{Mod}_{\mathcal{O}_Z}(Z) \to \mathrm{Mod}_{\mathcal{O}_X}(X).$$

Proof. See exercise 2.8. □

Quasi-coherent sheaves.

Let R be a ring and let M be an R-module. We define a sheaf $\widetilde{M}$ on $\mathrm{Spec}(R)$ by the recipe

$$\begin{aligned} \widetilde{M}(U) := \{ s : U \to \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} M_{\mathfrak{p}} \mid & \text{ for all } \mathfrak{p} \in U \text{ we have } s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ & \text{and for all } \mathfrak{p} \in U \text{ there is } a \in M, r \in R \text{ and } V \in \mathrm{Top}(U) \\ & \text{such that } D_r(R) \supseteq V \supseteq \{\mathfrak{p}\} \text{ and } s(\mathfrak{q}) = \frac{a}{r} \text{ for all } \mathfrak{q} \in V \} \end{aligned} \tag{19}$$

(notice that the definition (16) is the case $M = R$).

The sheaf $\widetilde{M}$ carries an obvious $\mathcal{O}_{\mathrm{Spec}(R)}$-module structure. Also, if $M \to N$ is a morphism of R-modules, there is an obvious associated morphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\widetilde{M} \to \widetilde{N}$. We have thus defined a functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Proposition 2.19. (a) *For all $r \in R$, we have a canonical isomorphism $\widetilde{M}(D_r(R)) \overset{(*)}{\simeq} M_r$. If we endow $\widetilde{M}(D_r(R))$ with its natural $\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R))$-module structure and M_r with its natural R_r-module structure, then the isomorphism $(*)$ is compatible with these module structures via the isomorphism of 2.5(a).*

(b) *If $t \in R$ and $t \in (r)$ then there is a commutative diagram*

$$\begin{array}{ccc} \widetilde{M}(D_r(R)) & \longrightarrow & \widetilde{M}(D_t(R)) \\ \downarrow \simeq & & \downarrow \simeq \\ M_r & \longrightarrow & M_t \end{array}$$

where the vertical isomorphisms come from (a). The horizontal morphisms are compatible with the various module structures in an obvious way.

(c) *There is a natural isomorphism $\widetilde{M}_{\mathfrak{p}} \simeq M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. This isomorphism fits in a commutative diagram*

$$\begin{array}{ccc} \widetilde{M}_{\mathfrak{p}} & \overset{\simeq}{\longrightarrow} & M_{\mathfrak{p}} \\ \uparrow & & \uparrow \\ \widetilde{M}(\mathrm{Spec}(R)) & \overset{\simeq}{\longrightarrow} & M \end{array}$$

Here the vertical morphisms are the natural ones and the lower horizontal one comes from (a).

Proof. The proof of this Proposition is similar to the proof of Proposition 2.5. We skip the details. □

Corollary 2.20. *The functor $\widetilde{\bullet}$ from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is fully faithful and exact.*

Proof. See Exercise 2.9. □

Let now X be a scheme.

Definition 2.21. *Let F be a sheaf on $\mathcal{O}_X$-modules. The sheaf F is said to be quasi-coherent (resp. coherent) if there is an open affine covering (U_i) of X, such that $F|_{U_i} \simeq \widetilde{F(U_i)}$ (resp. $F|_{U_i} \simeq \widetilde{F(U_i)}$ and $F(U_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-module).*

The full subcategory of $\mathrm{Mod}(X)$, which are quasi-coherent, will be denoted $\mathfrak{Q}coh(X)$.

Lemma 2.22. *Let $\phi : R \to T$ be a morphism of rings. Let M be a T-module. Then there is a natural isomorphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathrm{Spec}(\phi)_*(\widetilde{M}) \simeq \widetilde{M_0}$, where M_0 is M viewed as an R-module via ϕ.*

Proof. Notice that for all $r \in R$, there a natural isomorphisms of R_r-modules

$$\mathrm{Spec}(\phi)_*(\widetilde{M})(D_r(R)) = \widetilde{M}(\mathrm{Spec}(\phi)^{-1}(D_r(R)) = \widetilde{M}(D_{\phi(r)}(T)) \simeq M_{\phi(r)} \simeq M_{0,r},$$

which are compatible with restrictions $D_r(R) \supseteq D_{r'}(R)$ for $r' \in (r)$. This follows from Proposition 2.19 (b) and Exercise 2.10. Now the lemma follows from the fact that the sets $D_r(R)$ form a basis for the topology of $\mathrm{Spec}(R)$ and the fact that $\mathrm{Spec}(\phi)_*(\widetilde{M})$ and $\widetilde{M_0}$ are both sheaves. □

Proposition 2.23. *The definition of a quasi-coherent (resp. coherent) sheaf is independent of the open affine covering appearing in its definition.*

Proof. Let X be a scheme. Let F be a quasi-coherent sheaf on X. Let $U \subseteq X$. We claim first that $F|_U$ is also quasi-coherent. To show this, let $(U_i = \mathrm{Spec}(R_i))$ be an open affine covering of X such that $F|_{U_i} \simeq \widetilde{M_i}$ for some R_i-module M_i. For each i, let (U_{ij}) be a covering of $U \cap U_i$ by open subsets of U_i of the form $D_{f_j}(R_i)$. Then by Proposition 2.19 (b), we have $F|_{U_{ij}} \simeq \widetilde{M_{i,f_j}}$ on $D_{f_j}(R_i) = \mathrm{Spec}(R_{i,f_j})$. Since the family of all the (U_{ij}) form an open affine covering of U, this proves the claim.

Let now (U_i) and (V_j) be two affine coverings of X. The conclusion of the Proposition is that a sheaf F in $\mathcal{O}_X$-modules is quasi-coherent with respect to (U_i) if and only if it is quasi-coherent with respect to (V_j). Using the above claim and looking for each i at the covering $(U_i \cap V_j)$ of U_i, we see that to prove the Proposition, is it sufficient to prove the proposition in the situation where the covering (U_i) consists of a single affine scheme. In other words, it is sufficient to prove that if $X = \mathrm{Spec}(R)$ is affine and F is quasi-coherent on X with respect to an open affine covering $(\lambda_j : V_j = \mathrm{Spec}(T_j) \to X)$ of X, then $F \simeq \widetilde{M}$ for some R-module M (in other words, F is in the essential image of the functor $\widetilde{\bullet}$). Each V_j can be covered by open subsets of the form $D_{f_{kj}}(R)$. Hence we may assume that $V_j = \mathrm{Spec}(R_{f_j})$ for some $f_j \in R$ and that the covering (V_j) is finite. Notice that in this situation any finite intersection of V_j is also affine. Now consider the Cech complex associated with the covering (V_j) of X. By Lemma 2.22 and the previous remark, the terms of this complex are all $\mathcal{O}_X$-modules, which are in the essential image of the functor $\widetilde{\bullet}$. Looking at the two first terms of the Cech complex, we see that it exhibits F as the kernel of a morphism between two sheaves, which are in the essential image of $\widetilde{\bullet}$. We conclude by appealing to Corollary 2.20. □

Lemma 2.24. *Let R be a noetherian ring and let M be an R-module. Let $\mathfrak{a}$ be an ideal of R. There is an isomorphism*

$$\widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a})) \simeq \varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M),$$

which is natural in M.

Proof. (partial) Let $f_1, \dots, f_k \in \mathfrak{a}$ be a set of generators of $\mathfrak{a}$ (such a set exists because R is noetherian). Define

$$\mathfrak{a}_n := (f_1^n, \dots, f_k^n)$$

For any $n \geqslant 0$, there is a natural map

$$\mathrm{Hom}_R(\mathfrak{a}^n, M) \to \mathrm{Hom}_R(\mathfrak{a}_n, M)$$

and we leave it to the reader to show that these maps induce an isomorphism

$$\varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M) \simeq \varinjlim_n \mathrm{Hom}_R(\mathfrak{a}_n, M)$$

Note that by Lemma 2.1, we have

$$\mathrm{Spec}(R)\backslash V(\mathfrak{a}) = \bigcup_i D_{f_i}$$

Now let $\beta_{n_0} \in \mathrm{Hom}_R(\mathfrak{a}_{n_0}, M)$ be a representative of an element $\beta \in \varinjlim_n \mathrm{Hom}_R(\mathfrak{a}_n, M)$. For each i, let

$$\beta_{n_0}(f_i^{n_0})/f_i^{n_0} \in \widetilde{M}(D_{f_i}) = M_{f_i}.$$

We assert that this element only depends on β and that the $\beta_{n_0}(f_i^{n_0})/f_i^{n_0}$ glue to give an element $\widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a}))$.

We first prove the first assertion. Let $n_1 > n_0$ and suppose that $\beta_{n_1} \in \mathrm{Hom}_R(\mathfrak{a}_{n_1}, M)$ also represents β. We compute

$$\beta_{n_0}(f_i^{n_0})/f_i^{n_0} = \beta_{n_0}(f_i^{n_0} \cdot f_i^{n_1-n_0})/f_i^{n_0+n_1-n_0} = \beta_{n_1}(f_i^{n_1})/f_i^{n_1}$$

whence the assertion.

Now to the second assertion. By the sheaf property, in order to glue the $\beta_{n_0}(f_i^{n_0})/f_i^{n_0}$ into an element $\widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a}))$, we have to prove that the image of $\beta_{n_0}(f_i^{n_0})/f_i^{n_0}$ in $M_{f_if_j}$ is equal to the image of $\beta_{n_0}(f_j^{n_0})/f_j^{n_0}$ in $M_{f_if_j}$. We compute

$$\text{image of } \beta_{n_0}(f_j^{n_0})/f_j^{n_0} \text{ in } M_{f_if_j} = \beta_{n_0}(f_j^{n_0} \cdot f_i^{n_0})/(f_j^{n_0} \cdot f_i^{n_0}) = \text{image of } \beta_{n_0}(f_i^{n_0})/f_i^{n_0} \text{ in } M_{f_if_j}$$

proving the second assertion.

We have thus defined a map $\iota : \varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M) \to \widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a}))$ and it can be verified easily that this map is a map of abelian groups and that it is natural in M. To conclude the proof of the lemma, it suffices to show that ι is an isomorphism.

For any n, let ${}_{\mathfrak{a}^n}M$ be the submodule of elements in M, whose annihilator contains $\mathfrak{a}^n$.

Assume first that ${}_{\mathfrak{a}^n}M = 0$ for all n.

We shall provide an map $\lambda : \widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a})) \to \varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M)$ inverse to ι.

Consider a family of $m_i/f_i^l \in M_{f_i}$ such that the image of m_i/f_i^l in $M_{f_if_j}$ is equal to the image of m_j/f_j^l in $M_{f_if_j}$ for all i, j. This is a concrete description of an element $s \in \widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a}))$. This implies that for all i, j we have

$$(f_if_j)^{l_0} f_j^l m_i = (f_if_j)^{l_0} f_i^l m_j \tag{20}$$

for some $l_0 \geqslant 0$, which we may wrog assume independent of i, j.

We define $\lambda(s)$ as the element of $\varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M)$ represented by the morphism

$$\lambda_{l+l_0}(s) : \mathfrak{a}^{l+l_0} \to M$$

of R-modules, which is defined by the formula

$$(\lambda_{l+l_0}(s))\big(\sum_i b_i f_i^{l+l_0}\big) = \sum_i b_i m_i f_i^{l_0}.$$

This is well-defined, for if $\sum_i b_i f_i^{l+l_0} = 0$ then for all j, we have

$$(\lambda_{l+l_0}(s))\big(\sum_i b_i f_i^{l+l_0}\big) f_j^{l+l_0} = (\sum_i b_i m_i f_i^{l_0}) f_j^{l+l_0} = (\sum_i b_i f_i^{l+l_0}) f_j^{l_0} m_j = 0$$

by the equations (20) and thus $(\lambda_{l+l_0}(s))\big(\sum_i b_i f_i^{l+l_0}\big) = 0$ since ${}_{\mathfrak{a}_n}M = 0$. It is straightforward to verify that $\lambda \circ \iota = \mathrm{Id}$ and $\iota \circ \lambda = \mathrm{Id}$. For the general case where ${}_{\mathfrak{a}_n}M \neq 0$ for some n, we refer to [Har77, Lemma II.3.2 and after]. □

Let X be a scheme. A subsheaf of $\mathcal{O}_X$ is called *a sheaf of ideals* on X. Let J be a quasi-coherent sheaf of ideals on X.

Proposition 2.25. *There exists a closed immersion $(z, z^{\#}) : Z \to X$ such that $J = \ker(z^{\#})$. This immersion is unique up to unique isomorphism over X.*

Proof. See Exercise 2.11. □

Permanence properties of quasi-coherent sheaves.

Let X be a ringed space and let (F_i) be a family of $\mathcal{O}_X$-modules on X. We write $\bigoplus_i F_i$ for the sheaf generated by the presheaf in $\mathcal{O}_X$-modules on X sending $U \in \mathrm{Top}(X)$ to $\bigoplus_i F_i(U)$.

Let I be an index set. A family of $\mathcal{O}_X$-modules may be viewed as a functor $I \to \mathrm{Mod}(X)$, where I is viewed as a category with no arrows. The category of these functors is often denoted by $\mathrm{Mod}(X)^I$. The direct sum $\bigoplus_i(\bullet)$ and the product $\prod_i(\bullet)$ can be viewed as functors $\mathrm{Mod}(X)^I \to \mathrm{Mod}(X)$.

Lemma 2.26. *For any object (F_i) of $\mathrm{Mod}(X)^I$ and any object G in $\mathrm{Mod}(X)$, there is a canonical isomorphism*

$$\mathrm{Mor}(\bigoplus_i F_i, G) \simeq \prod_i \mathrm{Mor}(F_i, G)$$

which is natural in (F_i) and G.

Proof. See Exercise 2.12. □

In categorical terms, Lemma 2.26 says that the direct sum is a categorical coproduct in the category $\mathrm{Mod}(X)$.

Lemma 2.27. *Let X be a scheme and let (F_i) be a family of quasi-coherent sheaves on X. Then $\bigoplus_i F_i$ is quasi-coherent.*

Proof. Let R be a ring and (M_i) be a family of R-modules. If $r \in R$, there is a functorial isomorphism $(\bigoplus_i M_i)_r \simeq \bigoplus_i M_{i,r}$ (look at the definition of localisation in [AM69, chap. III]). The Lemma follows from this and Proposition 2.19. □

Remark 2.28. A formal consequence of the last two lemmata is the following fact. Let R be a ring and let (M_i) be a family of R-modules. Then there is a functorial isomorphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

$$\widetilde{(\bigoplus_i M_i)} \simeq \bigoplus_i \widetilde{M_i}$$

Proposition 2.29. *Let $\phi : R \to T$ be a morphism of rings and let M be an R-module. Then $\mathrm{Spec}(\phi)^*(\widetilde{M})$ is a quasi-coherent sheaf.*

Proof. First notice the following fact. Let $(X, \mathcal{O}_X) := (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. Let G be a $\mathcal{O}_X$-module. Then G is quasi-coherent if and only if there exist index sets I and J and exact sequence of $\mathcal{O}_X$-modules

$$\bigoplus_{i \in I} \mathcal{O}_X \to \bigoplus_{j \in J} \mathcal{O}_X \to G \to 0 \tag{21}$$

To see this, suppose that a $\mathcal{O}_X$-module G has a presentation (21).Then by Corollary 2.20 and Remark 2.28 above, we conclude that G is quasi-coherent. On the other hand, if $G = \widetilde{M}$ for some R-module then we may choose a surjection $u : \bigoplus_{j\in J} R \to M$ and a surjection $\bigoplus_{i\in I} R \to \ker(u)$. Applying the functor $\widetilde{\bullet}$, while taking into account Remark 2.28, we obtain a presentation (21).

Let $(Y, \mathcal{O}_Y) := (\mathrm{Spec}(T), \mathcal{O}_{\mathrm{Spec}(T)})$. In view of the above fact and the fact that $\mathrm{Spec}(\phi)^*$ is right exact, we see that we are reduced to prove that there is an isomorphism

$$f^*(\bigoplus_i \mathcal{O}_X) \simeq \bigoplus_i \mathcal{O}_Y \tag{22}$$

To show this, first notice that there is an isomorphism $f^*(\mathcal{O}_X) \simeq \mathcal{O}_Y$. For this notice that by Lemma 2.18, we have canonical isomorphisms for any $\mathcal{O}_Y$-module

$$\mathrm{Mor}_{\mathrm{Mod}(Y)}(f^*(\mathcal{O}_X), G) \simeq \mathrm{Mor}_{\mathrm{Mod}(X)}(\mathcal{O}_X, f_*(G)) \simeq f_*(G)(X) \simeq G(Y) \simeq \mathrm{Mor}_{\mathrm{Mod}(Y)}(\mathcal{O}_Y, G)$$

and thus $f^*(\mathcal{O}_X)$ and $\mathcal{O}_Y$ represent the same covariant functor. We conclude by appealing to Yoneda's lemma. To prove that there is an isomorphism (22), we notice that there are functorial isomorphisms

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Mod}(Y)}(f^*(\bigoplus_i \mathcal{O}_X), G) &\simeq \mathrm{Mor}_{\mathrm{Mod}(X)}(\bigoplus_i \mathcal{O}_X, f_*(G)) \\ &\simeq \prod_i \mathrm{Mor}_{\mathrm{Mod}(X)}(\mathcal{O}_X, f_*(G)) = \prod_i f_*(G)(X) = \prod_i G(Y) \end{aligned}$$

where we have used Lemma 2.26. Thus, we have functorial isomorphisms

$$\mathrm{Mor}_{\mathrm{Mod}(Y)}(\bigoplus_i f^*(\mathcal{O}_X), G) \simeq \mathrm{Mor}_{\mathrm{Mod}(Y)}(\bigoplus_i \mathcal{O}_Y, G) = \prod_i G(Y)$$

and thus again $\bigoplus_i \mathcal{O}_Y$ and $f^*(\bigoplus_i \mathcal{O}_X)$ represent the same covariant functor and must thus be isomorphic. □

Corollary 2.30. *In the situation of Proposition 2.29, there is a functorial isomorphism*

$$\mathrm{Spec}(\phi)^*(\widetilde{M}) \simeq \widetilde{M \otimes_R T}.$$

Proof. If we combine Proposition 2.29 with Lemma 2.18, Lemma 2.20 and Lemma 2.22, we obtain by restriction a pair of adjoint functors $f^* : \mathfrak{Q}coh(X) \to \mathfrak{Q}coh(Y)$ and $f_* : \mathfrak{Q}coh(Y) \to \mathfrak{Q}coh(X)$ (we abused the notation slightly). Now by Proposition 2.23, the category $\mathfrak{Q}coh(Y)$ is equivalent to the category of T-modules (and similarly for X). Furthermore, by Lemma 2.22, in terms of T-modules, the functor f_* send a T-module on the same module, viewed as an R-module via ϕ. Hence the conclusion of the corollary follows from the fact that there is a functorial isomorphism

$$\mathrm{Mor}_R(N, M) \simeq \mathrm{Mor}_T(N \otimes_R T, M)$$

for any R-module N and T-module M. The proof of this fact is left as an exercise (see Exercise 2.13). □

Corollary 2.31. *Let $f : X \to Y$ be a morphism of schemes. Let F be quasi-coherent sheaf on Y. Then $f^*(F)$ is also quasi-coherent.*

Proposition 2.32. *Let $f : X \to Y$ be a morphism of schemes. Suppose that X is noetherian. Let F be a quasi-coherent $\mathcal{O}_X$-module. Then $f_*(F)$ is also quasi-coherent.*

Proof. We may assume wrog that Y is affine. Let (U_i) be a finite open affine cover of X (this exists because X is quasi-compact) and for all i, j let U_{ijk} be a finite open affine cover of $U_i \cap U_j$ indexed by k (this exists by Lemma 2.12). Looking at the beginning of the Cech complex and using the fact that f_* is left exact as a functor from $\mathrm{Mod}_{\mathcal{O}_X}(X)$ to $\mathrm{Mod}_{\mathcal{O}_Y}(Y)$, we see that there is an exact sequence

$$0 \to f_*(F) \to \bigoplus_i f_*(F|_{U_i}) \to \bigoplus_{i,j,k} f_*(F|_{U_{ijk}})$$

From this and Corollary 2.20, we see that is sufficient to prove Proposition 2.32 under the assumption that X is also affine. In this case, it follows from Lemma 2.22. $\square$

Relative cohomology of quasi-coherent sheaves.

Let $f : X \to Y$ be a morphism of ringed spaces. Let F be an $\mathcal{O}_X$-module. It is proven in Exercise 2.1 that $\mathrm{Mod}(X)$ has enough injectives and that the injective objects of $\mathrm{Mod}(X)$ are flasque. Thus, taking into account Proposition 1.22 and Exercise 1.5, the abelian sheaves $R^k f_*(F)$ are naturally $\mathcal{O}_Y$-modules, which can be obtained by computing the derived functors of the direct image functor $\mathrm{Mod}(X) \to \mathrm{Mod}(Y)$.

Complement 2.33. Let the terminology of Corollary 1.24 hold. Suppose that X, Y and Z are ringed spaces and that f, g are morphisms of ringed spaces. Suppose that $F^\bullet$ is a bounded sequence of $\mathcal{O}_X$-modules. Then the Leray spectral sequence respects the various $\mathcal{O}_Z$-module structures. This follows immediately from the previous remarks.

Proposition 2.34. *Let $f : X \to Y$ be a morphism of schemes. Let F be a quasi-coherent sheaf on X. Suppose that X is noetherian. Then the $\mathcal{O}_Y$-module $R^k f_*(F)$ is also quasi-coherent.*

Proof. Let $(j_i : U_i \to X)$ be a finite open affine cover of X (recall that X is quasi-compact by definition, because it is noetherian). By the sheaf property, we have an exact sequence

$$0 \to F \to \bigoplus_i j_{i,*}(F|_{U_i}).$$

On the other hand the quasi-coherent $\mathcal{O}_{U_i}$-module $F|_{U_i}$ can be embedded in the $\mathcal{O}_{U_i}$-module associated with an injective $\Gamma(U_i, \mathcal{O}_{U_i})$-module I_i. Now the $\mathcal{O}_{U_i}$-module $\widetilde{I}_i$ is flasque by Lemma 2.24. Hence the sheaf $\bigoplus_i j_{i,*}(\widetilde{I}_i)$ is also flasque and composing morphisms we obtain an exact sequence

$$0 \to F \to \bigoplus_i j_{i,*}(I_i)$$

Repeating this process with the sheaf $\bigoplus_i j_{i,*}(I_i)/F$ (which is quasi-coherent by Corollary 2.20) and continuing this way we obtain a sequence

$$0 \to F \to F^0 \to F^1 \to \dots$$

where all the F^k are quasi-coherent and flasque. By Exercise 1.5 and Complement 1.23, we see that

$$R^k f_*(F) \simeq \mathcal{H}^k(f_*(F_\bullet))$$

and $\mathcal{H}^k(f_*(F_\bullet))$ is quasi-coherent by Proposition 2.32 and Corollary 2.20. $\square$

2.4 Cohomological characterisation of affine schemes.

Proposition 2.35. *Let X be a noetherian affine scheme and let F be a quasi-coherent sheaf on X. Then $H^k(X,F) = 0$ for all $k > 0$.*

Proof. Suppose $X = \mathrm{Spec}(R)$. If I is an injective R-module, it follows from Lemma 2.24 that the sheaf $\widetilde{I}$ is flasque. The proposition follows from this, Proposition 1.22, Exercise 1.13 and Corollary 2.20. □

Lemma 2.36. *Let $X \simeq (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme and let F be a quasi-coherent sheaf on X. Let $f \in R$ and let $s_0 \in F(X)$ and $s_1 \in F(D_f(R))$.*

(a) If $s_0|_{D_f(R)} = 0$ then for some $n \geqslant 0$, we have $f^n \cdot s_0 = 0$.

(b) For some $n \geqslant 0$, we have $f^n \cdot s_1 \in F(D_f(R) \to X)(F(X))$.

Proof. Let $F \simeq \widetilde{M}$ for some R-module M (see Proposition 2.23). Identify s_0 with the corresponding element of $M \simeq F(X)$. The condition that $s_0|_{D_f(R)} = 0$ corresponds to the condition that the image of s_0 in M_f vanishes (use Proposition 2.19 (b) and Corollary 2.20 or simply the definition of $\widetilde{M}$) . By the definition of localisation, the image of s_0 in M_f vanishes if and only if there is $n \geqslant 0$ such that $f^n \cdot s_0 = 0$ in M. This proves (a) under the above assumption. The proof of (b) under the same assumption is similar. □

Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$. We define

$$X_f := \{x \in X \mid f_x \notin \text{maximal ideal of } \mathcal{O}_{X,x}\}$$

If $X = \mathrm{Spec}(R)$, then $X_f = D_f(R)$, which is open in X. From this, one can see that X_f is open for any scheme X.

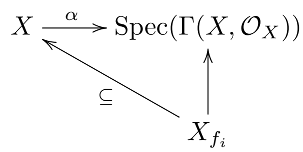
Lemma 2.37. *Let X be a noetherian scheme and let $f \in \Gamma(X, \mathcal{O}_X)$. Then there is a natural isomorphism $\Gamma(X, \mathcal{O}_X)_f \stackrel{\sim}{\to} \Gamma(X_f, \mathcal{O}_{X_f})$.*

Proof. There is a natural restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_f, \mathcal{O}_{X_f})$. Now notice that $f|_{X_f}$ is a unit in the ring $\Gamma(X_f, \mathcal{O}_{X_f})$. This follows from the local description of X_f given above, which shows that $f|_{U_f}$ is a unit for any open affine subscheme $U \subseteq X$. Hence the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_f, \mathcal{O}_{X_f})$ extends uniquely to a map $\rho_{X,f} : \Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_{X_f})$. Now take a finite open affine covering (U_i) of X and choose a finite open affine covering U_{ijk} of each $U_i \cap U_j$. All this is possible by the noetherian hypothesis. Let $t \in \Gamma(X_f, \mathcal{O}_{X_f})$. By Lemma 2.36 (b), for each i, there is an $n \geqslant 0$ such that the element $f^n \cdot t|_{U_i \cap X_f}$ is the restriction of some element $\lambda_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. This n can be taken independent of i, since there is only a finite number of indices. Furthermore, for each triple of indices i, j, k, the restriction of $\lambda_i - \lambda_j$ to U_{ijk} is annihilated by some power of f, by Lemma 2.36 (a). Hence, for a sufficiently large k, the restriction of $f^k \cdot \lambda_i - f^k \cdot \lambda_j$ to $U_i \cap U_j$ will vanish and hence by the sheaf property, the $f^k \cdot \lambda_i$ glue to an element $\lambda \in \Gamma(X, \mathcal{O}_X)$ such that $\rho_{X,f}(\lambda) = f^{k+n} \cdot t$. This proves that $\rho_{X,f}$ is surjective. The proof of injectivity is similar and will be skipped. □

Complement 2.38. Suppose that a scheme X has a finite open affine covering (U_i) such that that all the intersections $U_i \cap U_j$ have finite open affine coverings. Then the proof shows that Lemma 2.37 holds for X, even without the noetherian hypothesis.

Corollary 2.39. *Let X be a noetherian scheme and let $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ be such that $(f_1, \dots, f_n) = \Gamma(X, \mathcal{O}_X)$. If the open subschemes X_{f_i} are all affine, then X is affine.*

Proof. We have for all i a commutative diagram



where the straight vertical arrow comes from Lemma 2.37 and the horizontal one comes from Lemma 2.17. The verification of the commutativity of the diagram can be proven by reduction to the case where X is affine and is left to the reader. Since the morphisms $X_f \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ are open immersions by Lemma 2.37, we see that the restriction of the morphism α to the image of X_{f_i} in $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is an isomorphism. Since the images of the X_{f_i} cover $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ by assumption, we see that α is an isomorphism. □

Lemma 2.40. *Let X be a scheme and let $C_0 \subseteq X$ be a closed subset. Then there is a quasi-coherent ideal I_{C_0} in X, such that the image of the closed immersion $C \to X$ associated by Lemma 2.25 with I_{C_0} is C_0 and such that C is reduced. The quasi-coherent ideal I_{C_0} is uniquely determined by these requirements.*

Proof. Exercise 2.16. □

One often writes $C_{0,\mathrm{red}} \to X$ for the closed immersion whose existence is asserted in the lemma.

Remark 2.41. *(important!)* The closed immersion $X_{0,\mathrm{red}} \to X$ is surjective (prove this!) and is thus a homeomorphism.

Theorem 2.42 (Serre)**.** *Let X be a noetherian scheme and suppose that for all coherent sheaves F on X, we have $H^1(X, F) = 0$ for all $k > 0$. Then X is an affine scheme.*

Proof. Let P be a closed point in X. This exists by Exercise 2.15. Let U be an open affine neighbourhood of P and let Y be the complement of U in X. We view P, Y and $P \cup Y$ as reduced closed subschemes of X, via Lemma 2.40. Let I_P, I_Y and $I_{P\cup Y}$ be the corresponding quasi-coherent sheaves of ideals. Note that we have canonically $\mathcal{O}_P(P) \simeq \kappa(P)$ and that this isomorphism describes the sheaf $\mathcal{O}_P$ entirely.

By construction, we have an exact sequence

$$0 \to I_{Y\cup P} \to I_Y \to \kappa(P) \to 0$$

where $\kappa(P)$ denotes the direct image of $\mathcal{O}_P$ by the closed immersion $P \to X$. Applying Theorem 1.3 to this sequence and to the functor $\Gamma(X, \bullet)$, we obtain an exact sequence

$$\Gamma(X, I_Y) \to \Gamma(X, \kappa(P)) \to H^1(X, I_{Y\cup P})$$

and since by assumption $H^1(X, I_{Y\cup P}) = 0$, we get a surjection $\Gamma(X, I_Y) \to \Gamma(X, \kappa(P))$. Let $f \in \Gamma(X, I_Y)$ be such that the image of f in $\Gamma(X, \kappa(P)) \simeq \kappa(P)$ is 1. We view f as an element of $\Gamma(X, \mathcal{O}_X)$ via the natural inclusion $\Gamma(X, I_Y) \to \Gamma(X, \mathcal{O}_X)$.

By construction, we have that $P \in X_f$ and also that $X_f \subseteq U$. In particular, X_f is affine, because it corresponds to a basic open set in U. If $X \neq X_f$, we now repeat this reasoning for a closed point P_2 in $X \backslash X_f$

(this is possible because $X \backslash X_f$ is quasi-compact, since X is noetherian) and obtain an affine neighbourhood U_2 of P_2 and $f_2 \in \Gamma(X, \mathcal{O}_X)$ such that $P_2 \in X_{f_2}$ and X_{f_2} is affine and we repeat it for $P_3 \in X \backslash X_f \cup X_{f_2}$ etc. The sequence of the X_{f_i} must stop after a finite number of steps, and thus cover X, because X is a noetherian topological space (by Lemma 2.12).

We can thus exhibit a finite sequence $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that X_{f_i} is affine for all i and such that the X_{f_i} cover X. By Lemma 2.39, we shall be able to conclude if we can show that the f_i generate $\Gamma(X, \mathcal{O}_X)$. To see this, consider the morphism of sheaves

$$\bigoplus_{i=1}^{n} \mathcal{O}_X \to \mathcal{O}_X$$

sending local sections $(s_1, \dots, s_n)$ to $\sum_i f_i \cdot s_i$. This morphism is surjective, because the X_{f_i} cover X (check this surjectivity locally and use Lemma 2.1). Applying Theorem 1.3, Corollary 2.20 and using the assumptions we obtain a surjection

$$\bigoplus_{i=1}^{n} \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X).$$

In other words, the f_i generate $\Gamma(X, \mathcal{O}_X)$. □

2.5 Exercises

Exercise 2.1. *Let X be a ringed space. Prove that $\mathrm{Mod}(X)$ has enough injectives. Prove that injective objects in $\mathrm{Mod}(X)$ are flasque as abelian sheaves.*

Exercise 2.2. *Prove Lemma 2.1.*

Exercise 2.3. *Prove Lemma 2.3.*

Exercise 2.4. *Let R be a ring and $\mathfrak{p}$ a prime ideal of R. Show that there is a natural isomorphism*

$$\varinjlim_{r \in R; r \notin \mathfrak{p}} R_r \simeq R_{\mathfrak{p}}$$

Here the arrows in the inductive system are defined as follows. If r' is a multiple of r then the arrow is the natural map $R_r \to R_{r'}$. Otherwise there is no arrow.

Exercise 2.5. *(see the beginning of Subsection 2.2) Let R be a ring. Show that R is noetherian if and only if the submodules of any finitely generated R-module M are all finitely generated as well. Show that R is noetherian if and only if any ascending sequence of ideals of R becomes stationary after a finite number of steps. Show that a localisation of a noetherian ring is noetherian.*

Exercise 2.6. *Show that if a morphism is affine (resp. locally of finite type, resp. quasi-compact) with respect to a certain open affine covering then it is affine (resp. locally of finite type, resp. quasi-compact) with respect to any open affine covering.*

Exercise 2.7. *Prove Proposition 2.13. Hint: proceed as in the proof of Proposition 1.17.*

Exercise 2.8. *Prove Lemma 2.18.*

Exercise 2.9. *Prove Corollary 2.20.*

Exercise 2.10. *Let $\phi : R \to T$ be a morphism of rings and let M be a T-module. Let $S \subseteq R$ be a multiplicative set. Let M_0 be M viewed as an R-module. Then there is a natural isomorphism of R_S-modules $M_{0,S} \simeq M_{\phi(S)}$ and this morphism is compatible with inclusions of $S' \subseteq S$ of multiplicative sets.*

Exercise 2.11. *Prove Proposition 2.25.*

Exercise 2.12. *Prove Lemma 2.26. Hint: Consider the case of X =(a point) first.*

Exercise 2.13. *Let $\phi : R \to T$ be a morphism of rings. Let N be an R-module and M a T-module. Show that there is a functorial isomorphism*

$$\mathrm{Mor}_R(N, M) \simeq \mathrm{Mor}_T(N \otimes_R T, M)$$

where in the expression $\mathrm{Mor}_R(N, M)$, M is viewed as an R-module via ϕ.

Exercise 2.14. *Prove Lemma 2.17.*

Exercise 2.15. *Let X be a quasi-compact scheme. Prove that X has a closed point.*

Exercise 2.16. *Prove Lemma 2.40.*

Exercise 2.17. *Prove that a noetherian topological space is quasi-compact.*

Exercise 2.18. *Prove Lemma 2.15.*

Exercise 2.19. *Let $\phi : R \to T$ be a morphism of rings. Prove that the corresponding morphism of schemes $(\mathrm{Spec}(\phi), \phi^{\#}) : (\mathrm{Spec}(T), \mathcal{O}_{\mathrm{Spec}(T)}) \to (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a closed immersion if and only if ϕ is surjective.*

Exercise 2.20. *Show that the composition of two closed immersions is a closed immersion. Same for morphisms of finite type (resp. morphisms locally of finite type, resp. open immersions, quasi-compact morphisms).*

Exercise 2.21. *Show that a scheme is integral if and only it has a covering by open affine scheme $\mathrm{Spec}\, R_i$, where R_i is a domain. Show that a scheme is reduced if and only it has a covering by open affine scheme $\mathrm{Spec}\, R_i$, where R_i has no non vanishing nilpotent elements.*

3 Projective spaces

3.1 Affine spaces

Let $r \geqslant 0$. Consider the functor $\underline{\mathbb{A}}^r$ from **Schemes** to **Sets**, which associates with a scheme S the set of morphisms

$$\phi : \bigoplus_{k=1}^{r} \mathcal{O}_S \to \mathcal{O}_S$$

Lemma 3.1. *Let X be a scheme. The restriction of the functor $\underline{\mathbb{A}}^r$ to $\mathrm{Top}(X)$ is a sheaf of sets.*

Proof. See Exercise 3.6. □

Lemma 3.2. *Let X, S be schemes. Let h_S : **Schemes** $\to$ **Sets** be the functor $\mathrm{Mor}(\bullet, S)$. Then the restriction of h_S to $\mathrm{Top}(X)$ is a sheaf of sets.*

Proof. This is a formal consequence of Proposition 2.13. □

Proposition-Definition 3.3. *$\underline{\mathbb{A}}^r$ is representable by the scheme*

$$\mathbb{A}^r := \operatorname{Spec}(\mathbb{Z}[X_1, \ldots, X_r])$$

called the affine space of relative dimension r.

Proof. In view of Lemmata 3.1 and 3.2, it is sufficient to construct an isomorphism between the restriction of the functor $h_{\mathbb{A}^r}$ to Aff and the restriction of the functor $\underline{\mathbb{A}}^r$ to Aff. In the language of rings, we would like to provide a natural isomorphism between the following two functors. The restriction $h_{\mathbb{A}^r}|_{\mathrm{Aff}}$ of $h_{\mathbb{A}^r}$ to Aff in the language of rings is the functor

$$R \mapsto \operatorname{Mor}_{\mathrm{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$$

and the restriction $\underline{\mathbb{A}}^r|_{\mathrm{Aff}}$ of the functor $\underline{\mathbb{A}}^r$ to Aff in the language of rings is the functor

$$R \mapsto \operatorname{Mor}_{\mathbf{Sets}}(\{1, \ldots, r\}, R)$$

Now there is a natural transformation between $h_{\mathbb{A}^r}|_{\mathrm{Aff}}$ and $\underline{\mathbb{A}}^r|_{\mathrm{Aff}}$, which for every ring R maps $\operatorname{Mor}_{\mathrm{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$ to $\operatorname{Mor}_{\mathbf{Sets}}(\{1, \ldots, r\}, R)$, by sending

$$\phi \in \operatorname{Mor}_{\mathrm{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$$

to

$$\phi(X_\bullet).$$

This map is an isomorphism by the definition of polynomials (see [Lan02, II, par. 3, p. 97]). □

Let R be a ring and let $X \to \operatorname{Spec}(R)$ be a scheme over R. From the definitions, we see that to say that X is locally of finite type over R is the same as to say that there is an open covering (U_i) of X by affine open subschemes such that for each i, there is an $r(i) \in \mathbb{N}$ and a commutative diagram

$$\begin{array}{ccc} U_i & \longrightarrow & \mathbb{A}_R^{r(i)} := \mathbb{A}^{r(i)} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(R) \\ & \searrow & \downarrow \\ & & \operatorname{Spec}(R) \end{array}$$

where the vertical morphisms are the natural ones and the horizontal morphism is a closed immersion. These closed immersions are in general unrelated to each other and one may wonder what kind of compatibilities could be required. Projective spaces propose an answer to this question.

3.2 Projective spaces

Let $r \geqslant 0$. Consider the functor $\underline{\mathbb{P}}^r$ from **Schemes** to **Sets**, which associates with a scheme S the set of isomorphism classes of surjective morphisms

$$\phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}$$

where $\mathcal{L}$ is locally free of rank 1 (see Exercise 3.7 for this notion). Here a surjective morphism $\phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}$ is said to be isomorphic to a surjective morphism $\psi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{M}$ if there is an isomorphism $\iota : \mathcal{L} \simeq \mathcal{M}$ such that $\iota \circ \phi = \psi$.

A sheaf, which is locally free of rank one, is often called *a line bundle*.

Theorem 3.4. *The functor $\underline{\mathbb{P}}^r$ is representable by a scheme $\mathbb{P}^r$, which is integral and of finite type over* $\mathrm{Spec}(\mathbb{Z})$.

In particular $\mathbb{P}^r$ is noetherian. The scheme $\mathbb{P}^r$ is called *projective space of relative dimension* r.

Proof. Let K be the fraction field of the ring $\mathbb{Z}[X_0, \dots, X_r]$. Let $i, j, k \in \{0, \dots, r\}$. Define

$$R_i := \mathbb{Z}[\frac{X_0}{X_i}, \dots, \frac{X_r}{X_i}] \subseteq K$$

and

$$R_{ij} := R_{i, \frac{X_j}{X_i}} \subseteq K.$$

Here the notation $\mathbb{Z}[\frac{X_0}{X_i}, \dots, \frac{X_r}{X_i}]$ refers to the subring of K generated by the elements $\frac{X_0}{X_i}, \dots, \frac{X_r}{X_i}$. Notice that there is isomorphism between R_i and the abstract polynomial ring $\mathbb{Z}[Y_0, \dots \widehat{Y}_i, \dots Y_r]$ (where $\widehat{\bullet}$ means that the corresponding term is omitted), because $X_i/X_i = 1$.

Notice also that we have morphisms of R_i-algebras

$$R_{ij} \otimes_{R_i} R_{ik} \simeq R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}} \overset{\subseteq}{\to} K.$$

This morphism intertwines the natural R_{ij}-(resp. R_{ik})algebra structure of the first term with the R_{ij}-(resp. R_{ik})algebra of the second term arising from the inclusion $R_{ij} \subseteq R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}}$ (resp. $R_{ik} \subseteq R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}}$). Furthermore, it is easy to verify that we have the following set-theoretic relations between subsets of K:

$$R_i = R_{ii},\ R_{ij} = R_{ji},\ R_i \subseteq R_{ij},\ R_{jk} \subseteq R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}}$$

In view of these identities and the fact that any diagram of inclusions of subrings of K commutes, we see that the schemes $U_i = \mathrm{Spec}(R_i)$ and $U_{ij} = \mathrm{Spec}(R_{ij})$ together with the open immersions $U_{ij} \to U_i$ and the isomorphisms $U_{ij} \simeq U_{ji}$ coming from the corresponding inclusions of rings, define glueing data for schemes as described before Proposition 2.13. We thus obtain a scheme $\mathbb{P}^r$, which is integral and of finite type over $\mathbb{Z}$ by construction.

The scheme $\mathbb{P}^r$ carries a canonical line bundle $\mathcal{O}(1)$, which can be described using the following glueing data. Declare $\mathcal{O}(1)|_{U_i} = \mathcal{O}_{U_i}$ and let $\phi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}) = R_{ij}$ be given by X_i/X_j. We verify that $\phi_{ii} = 1$, $\phi_{ij} = \phi_{ji}^{-1}$ and that

$$\frac{X_j}{X_k} \cdot \frac{X_i}{X_j} = \frac{X_i}{X_k}$$

in $R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}}$, so that the ϕ_{ij} satisfy the conditions given before Proposition 1.17. We thus obtain an abelian sheaf on $\mathbb{P}^r$. The ϕ_{ij} are compatible with the $\mathcal{O}_{\mathbb{P}^r}$-module structure of $\mathcal{O}(1)$ on each U_i and so we see that $\mathcal{O}(1)$ is a $\mathcal{O}_{\mathbb{P}^r}$-module. Now $\mathbb{P}^r$ is by construction covered by the U_i, the U_i are affine and $\mathcal{O}(1)|_{U_i}$ is quasi-coherent (since it is trivial). Hence $\mathcal{O}(1)$ is a quasi-coherent sheaf and it is by construction locally free of rank 1.

For each $l = 0, \dots, n$, there is a canonical element $X_l \in \Gamma(\mathbb{P}^r, \mathcal{O}(1))$, such that

$$X_l|_{U_i} = X_l/X_i$$

via the identification $\mathcal{O}(1)|_{U_i} = \mathcal{O}_{U_i}$. Indeed, by Complement 1.18, this defines an element of $\Gamma(\mathbb{P}^r, \mathcal{O}(1))$, since

$$\phi_{ij}((X_l|_{U_i})|_{U_j}) = (X_l/X_i) \cdot (X_i/X_j) = X_l/X_j = (X_l|_{U_j})|_{U_i}.$$

Since $X_l|_{U_l}$ is a trivialisation of $\mathcal{O}(1)|_{U_l}$, we see that the collection of the X_l defines a surjection

$$\bigoplus_{k=0}^{r} \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}(1)$$

We shall now show that $\mathbb{P}^r$ represents $\underline{\mathbb{P}}^r$.

Let S be a scheme. If we are given a morphism $\phi : S \to \mathbb{P}^r$, we obtain by pull-back a surjection

$$\bigoplus_{k=0}^{r} \mathcal{O}_S \to \phi^*(\mathcal{O}(1)).$$

This construction provides a map $\mathbb{P}^r(S) \to \underline{\mathbb{P}}^r(S)$.

We wish to construct an inverse map $\underline{\mathbb{P}}^r(S) \to \mathbb{P}^r(S)$.

So let S be a scheme and let

$$\phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}$$

be a surjection of sheaves, where $\mathcal{L}$ is locally free of rank 1. We shall call $\sigma_0, \dots, \sigma_n$ the corresponding elements of $\Gamma(S, \mathcal{L})$. Let

$$S_{\sigma_i} := \{s \in S \,|\, \sigma_i \notin \mathfrak{m}_s \cdot \mathcal{L}_s\}$$

The set S_{σ_i} is open by the remark before Lemma 2.37 and because $\mathcal{L}$ is locally free. By Nakayama's lemma, the section $\sigma_i|_{S_{\sigma_i}}$ induces an isomorphism $\mathcal{O}_{S_{\sigma_i}} \simeq \mathcal{L}|_{S_{\sigma_i}}$. Identifying $\mathcal{L}|_{S_{\sigma_i}}$ with $\mathcal{O}_{S_{\sigma_i}}$ via this isomorphism, we obtain by restriction a morphism

$$\phi_{S_{\sigma_i}} : \bigoplus_{k=0}^{r} \mathcal{O}_{S_{\sigma_i}} \to \mathcal{O}_{S_{\sigma_i}}$$

whose k-th component is given σ_k/σ_i, where it is understood that σ_k/σ_i is a function on S_{σ_i} such that

$$(\sigma_k/\sigma_i) \cdot \sigma_i|_{S_{\sigma_i}} = \sigma_k|_{S_{\sigma_i}}.$$

By Proposition 3.3, $\phi_{S_{\sigma_i}}$ induces a morphism $f_i : S_{\sigma_i} \to U_i$, such that

$$(X_k/X_i) \circ f_i = \sigma_k/\sigma_i$$

(we abuse the notation here).

Now note that by construction, we have

$$f_i^{-1}(U_{ij}) = S_{\sigma_i} \cap S_{\sigma_j}$$

and similarly

$$f_j^{-1}(U_{ji}) = S_{\sigma_i} \cap S_{\sigma_j}.$$

Let $\psi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ be the canonical isomorphism (which is the identity in the above presentation). We compare $\psi_{ij} \circ f_i|_{S_{\sigma_i} \cap S_{\sigma_j}}$ and $f_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$. We compute

$$f_j|^*_{S_{\sigma_i} \cap S_{\sigma_j}}(X_k/X_j) = \sigma_k/\sigma_j$$

and

$$\psi_{ij} \circ f_i|^*_{S_{\sigma_i} \cap S_{\sigma_j}}(X_k/X_j) = \psi_{ij} \circ f_i|^*_{S_{\sigma_i} \cap S_{\sigma_j}}((X_k/X_i) \cdot (X_j/X_i)^{-1}) = (\sigma_k/\sigma_i) \cdot (\sigma_j/\sigma_i)^{-1} = \sigma_k/\sigma_j$$

so that $\psi_{ij} \circ f_i|_{S_{\sigma_i} \cap S_{\sigma_j}} = f_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$ by Proposition-Definition 3.3. Thus by Lemma 2.14 the family (f_i) of morphisms glue to a morphism $S \to \mathbb{P}^r$. So we have produced a map $\underline{\mathbb{P}}^r(S) \to \mathbb{P}^r(S)$.

We skip the easy verification of the fact that the two maps $\underline{\mathbb{P}}^r(S) \to \mathbb{P}^r(S)$ and $\mathbb{P}^r(S) \to \underline{\mathbb{P}}^r(S)$ are inverse to each other and functorial in S. □

3.3 Ample line bundles

Let S be a noetherian scheme.

A coherent F on S is said to be *generated by its global sections* or *globally generated* if there is a surjection

$$\bigoplus_{k=1}^{r_0} \mathcal{O}_S \to F$$

for some $r_0 \in \mathbb{N}$. The corresponding r_0 sections of F are then called *generating sections*.

Let now L be a line bundle on S.

Definition 3.5. *The line bundle L is ample if for any coherent sheaf F on S, there is $n_0 \in \mathbb{N}$ such that $F \otimes L^{\otimes n}$ is generated by its global sections for all $n \geqslant n_0$.*

Here $L^{\otimes n} := L \otimes L \otimes \cdots \otimes L$ (n-times).

Proposition 3.6. *The line bundle L is ample if and only if there is $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in \Gamma(S, L^{\otimes n})$ such that*

- *the schemes S_{σ_i} are affine;*
- *the schemes S_{σ_i} cover S.*

For the proof, we shall need the following

Lemma 3.7. *Let T_0 be a noetherian scheme and let M_0 be a coherent sheaf on T_0. Let L_0 be a line bundle on T_0. Let $f \in \Gamma(T_0, L_0)$ and let $s \in \Gamma(T_{0,f}, M_0)$. Then*

(a) *there is $n(s) \in \mathbb{N}$ such that $s \otimes f^{\otimes n(s)} \in \Gamma(T_{0,f}, M_0 \otimes L_0^{n(s)})$ extends to $\Gamma(T_0, M_0 \otimes L_0^{n(s)})$;*

(b) *if $s \in \Gamma(T_0, M_0)$ restricts to 0 in $\Gamma(T_{0,f}, M_0)$ then there is $n(s) \in \mathbb{N}$ such that $s \otimes f^{\otimes n(s)} \in \Gamma(T_0, M_0 \otimes L_0^{n(s)})$ vanishes.*

Proof. See Exercise 3.5. □

Proof. (of Proposition 3.6) We first prove the implication "$\Leftarrow$".

So suppose that there is $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in \Gamma(S, L^{\otimes n})$ such that (S_{σ_i}) is an open affine covering of S. Let F be a coherent sheaf on S. For each i, let $(\tau_{ij} \in \Gamma(S_{\sigma_i}, F|_{S_{\sigma_i}})$ be a finite family of generating sections of $F|_{S_{\sigma_i}}$. Such sections exist because S_{σ_i} is affine. By Lemma 3.7, there is $n \in \mathbb{N}$ such that for all i, the sections $\tau_{ij} \otimes \sigma_i^{\otimes n}|_{S_{\sigma_i}}$ extend to sections $\lambda_{ij} \in \Gamma(S, F \otimes L^{\otimes n})$. Now notice that the sections $\tau_{ij} \otimes \sigma_i^{\otimes n}|_{S_{\sigma_i}}$ are also generating sections of $F \otimes L^{\otimes n}|_{S_{\sigma_i}}$ because $L|_{S_{\sigma_i}}$ is by construction trivial. Hence the sections λ_{ij} (for all i, j) are generating sections of $\Gamma(S, F \otimes L^{\otimes n})$, since the S_{σ_i} cover S.

We now prove the implication "$\Rightarrow$".

Let $x \in S$. It is sufficient to show that there is $n(x) \in \mathbb{N}$ and $\sigma_x \in \Gamma(S, L^{\otimes n(x)})$ such that S_{σ_x} is affine and $x \in S_{\sigma_x}$. Let U be an affine neighbourhood of x such that $L|_U \simeq \mathcal{O}_U$ and let I be the ideal sheaf associated with $S \backslash U$ by Lemma 2.40. Let $\iota : (S\backslash U)_{\text{red}} \to S$ be the canonical closed immersion. Let $n(x) \in \mathbb{N}$ be such that there is $\bar{\sigma}_x \in \Gamma(S, I \otimes L^{\otimes n(x)})$ with $\bar{\sigma}_x \neq 0$.

Now consider the sequence of $\mathcal{O}_S$-modules

$$0 \to I \to \mathcal{O}_S \to \iota_*(\mathcal{O}_{(S\backslash U)_{\text{red}}}) \to 0 \tag{23}$$

and the sequence

$$0 \to I \otimes L^{\otimes n(x)} \to L^{\otimes n(x)} \to \iota_*(\mathcal{O}_{(S\backslash U)_{\text{red}}}) \otimes L^{\otimes n(x)} \to 0 \tag{24}$$

obtained by tensoring (23) by $L^{\otimes n(x)}$ (note that this sequence is exact because exactness can be checked locally and L is locally a trivial sheaf). If we apply the global sections functor $\Gamma(S, \bullet)$ to (24) we obtain a map $\Gamma(S, I \otimes L^{\otimes n(x)}) \to \Gamma(S, L^{\otimes n(x)})$. Let σ_x be the image of $\bar{\sigma}_x$ by this map. The section $\sigma_x \in \Gamma(S, L^{\otimes n(x)}$ vanishes on $S\backslash U$ by construction. Hence $S_{\sigma_x} \subseteq U$. Furthermore, since by assumption we have $L_U \simeq \mathcal{O}_U$, the set $S_{\sigma_x} \subseteq U$ is a basic open subset of the affine scheme U is thus also affine. □

Corollary 3.8. *The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^r$ is ample.*

Proof. Let X_i be the usual canonical section of $\mathcal{O}(1)$. The schemes $\mathbb{P}^r_{X_i}$ are by construction the affine scheme U_i in the standard open affine covering of $\mathbb{P}^r$. □

If R is a ring, we shall often write $\mathbb{P}^r_R$ for $\mathbb{P}^r \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(R)$.

Proposition 3.9. *Let $f : S \to \text{Spec}(R)$ be a morphism of finite type to the spectrum of a noetherian ring R. Let L be an ample line bundle on S. There is $n \in \mathbb{N}$ and $\sigma_0, \dots, \sigma_r \in \Gamma(S, L^{\otimes n})$ such that the σ_i generate $L^{\otimes n}$ and such that the corresponding morphism*

$$S \to \mathbb{P}^r_R$$

is a closed immersion into an open subset of $\mathbb{P}^r_R$.

Proof. We may wrog replace L by $L^{\otimes n}$ for some $n \geqslant 1$. By Proposition 3.6, we may thus assume that there is a finite family $(\sigma_i \in \Gamma(S, L))$ such that S_{σ_i} is affine and such that the S_{σ_i} cover S. For each i, let $\sigma_{ij} \in \Gamma(S_{\sigma_i}, L)$ be a family of sections, such that the functions $\sigma_{ij}/\sigma_i|_{S_{\sigma_i}}$ generate $\Gamma(S_{\sigma_i}, \mathcal{O}_{S_{\sigma_i}})$ as an R-algebra. For some $n \geqslant 0$, which can be taken independent of i, the sections $\sigma_i^{\otimes(n-1)}|_{S_{\sigma_i}} \otimes \sigma_{ij} \in \Gamma(S_{\sigma_i}, L^{\otimes n})$ extend to sections τ_{ij} of $L^{\otimes n}$ over S by Lemma 3.7. Now consider the disjoint union Σ of all the σ_i and all the τ_{ij} and choose an arbitrary identification $\phi : \{0, \dots, r\} \simeq \Sigma$. Since the σ_i already generate L, the set of sections Σ generates $L^{\otimes n}$ and via ϕ we obtain a $\text{Spec}(R)$-morphism $\iota : S \to \mathbb{P}^r_R$. This morphism is obtained by glueing together the morphisms

$$\iota_i : S_{\sigma_i} \to \text{Spec}(R[\frac{X_0}{X_{\phi^{-1}(\sigma_i)}}, \dots \frac{X_r}{X_{\phi^{-1}(\sigma_i)}}])$$

such that $\iota_i^*(\frac{X_k}{X_{\phi^{-1}(\sigma_i)}}) = \frac{\phi(k)|_{S_{\sigma_i}}}{\sigma_i|_{S_{\sigma_i}}}$. Since by construction the functions $\frac{\phi(k)|_{S_{\sigma_i}}}{\sigma_i|_{S_{\sigma_i}}}$ generate $\Gamma(S_{\sigma_i}, \mathcal{O}_{S_{\sigma_i}})$ as an R-algebra, and since S_{σ_i} is affine, we see that ι_i is a closed immersion. Thus ι is a closed immersion of S into the union in $\mathbb{P}^r_R$ of all the open affine subschemes $\text{Spec}(R[\frac{X_0}{X_{\phi^{-1}(\sigma_i)}}, \dots \frac{X_r}{X_{\phi^{-1}(\sigma_i)}}])$ (for all i). □

3.4 The cohomology of projective space

In this subsection, we shall investigate the cohomology of the tensor powers of the line bundle $\mathcal{O}(1)$ on projective space. We shall start with some preliminaries.

Tensor products of complexes.

Let R be a ring and let $(P^\bullet, d_P^\bullet)$ and $(Q^\bullet, d_Q^\bullet)$ be cochain complexes, which are bounded above. We may then form a double complex $(P^\bullet \otimes Q^\bullet)^{\bullet,\bullet}$. By definition, we have

$$(P^\bullet \otimes Q^\bullet)^{p,q} := P^p \otimes Q^q.$$

The horizontal differentials of $(P^\bullet \otimes Q^\bullet)^{\bullet,\bullet}$ are given by the formula

$$d'(a^p \otimes b^q) = d^p_{P^\bullet}(a^p) \otimes b^q$$

and the vertical differentials by the formula

$$d''(a^p \otimes b^q) = (-1)^q \cdot a^p \otimes d^q_{Q^\bullet}(b^q)$$

The total complex $\mathrm{Tot}((P^\bullet \otimes Q^\bullet)^{\bullet,\bullet})$ is written simply $P^\bullet \otimes Q^\bullet = (P^\bullet \otimes Q^\bullet)^\bullet$ and is called the tensor product of the complexes $P^\bullet$ and $Q^\bullet$. The symbols $(P^\bullet \otimes Q^\bullet)^{\bullet,\bullet}$ and $(P^\bullet \otimes Q^\bullet)^\bullet$ and are functorial in both arguments (we skip the verification).

Suppose now that $Q^\bullet$ is a complex of flat R-modules. The first page of the first spectral sequence of the double complex $(P^\bullet \otimes Q^\bullet)^{p,q}$ is then

$$E^{pq}_{I,1}(P^\bullet, Q^\bullet) = \mathcal{H}^q(P^\bullet \otimes Q^p) = \mathcal{H}^q(P^\bullet) \otimes Q^p \Rightarrow \mathcal{H}^{p+q}(P^\bullet)$$

We shall make use of this in the proof of the following lemma.

Lemma 3.10. *Let $\phi : P_1^\bullet \to P_2^\bullet$ be a quasi-isomorphism of cochain complexes of R-modules, where both complexes are supposed bounded above. Let $C^\bullet$ be another cochain complex of R-modules, which is bounded above. Suppose that either*

- *C^k is a flat R-module for all $k \in \mathbb{Z}$*
- *or P_1^k and P_2^k are flat R-modules for all $k \in \mathbb{Z}$.*

Then the morphism

$$\phi \otimes C^\bullet : P_1^\bullet \otimes C^\bullet \to P_2^\bullet \otimes C^\bullet$$

is a quasi-isomorphism.

Proof. The morphism $\phi \otimes C^\bullet$ arises from a morphism of double complexes

$$(P_1^\bullet \otimes C^\bullet)^{\bullet,\bullet} \to (P_2^\bullet \otimes C^\bullet)^{\bullet,\bullet}$$

which induces a morphism of spectral sequences

$$E^{pq}_{I,1}(P_1^\bullet, C^\bullet) \to E^{pq}_{I,1}(P_2^\bullet, C^\bullet)$$

Now by assumption this morphism of spectral sequences is an isomorphism on the first page. The lemma follows. □

The cohomology of finite intersections of basic open sets.

Let R be any ring and let I be a finite totally ordered index set. Let $(f_{i\in I} \in R)$ be a family of elements of R indexed by I. For all $p \geqslant 0$, let

$$C^p((f_{i\in I}), R) := \bigoplus_{i_0<i_1<\cdots<i_p} R_{f_{i_0}\cdots f_{i_p}}$$

and define a morphism of R-modules

$$d^p : C^p((f_{i\in I}), R) \to C^{p+1}((f_{i\in I}), R)$$

by the formula

$$\bigoplus_{i_0<i_1<\cdots<i_p} \alpha_{i_0\ldots i_p} \mapsto \sum_{k=0}^{p+1}(-1)^k \bigoplus_{i_0<i_1<\cdots<i_{p+1}} \alpha_{i_0\ldots\hat{i_k}\ldots i_{p+1}}$$

where $\alpha_{i_0\ldots\hat{i_k}\ldots i_{p+1}}$ is viewed as an element of $R_{f_{i_0}\cdots f_{i_{p+1}}}$. As usual the hat symbol $\hat{}$ signifies that the term under the hat is omitted. This gives a sequence of R-modules

$$C^0((f_{i\in I}), R) \stackrel{d^0}{\to} C^1((f_{i\in I}), R) \stackrel{d^1}{\to} \ldots \tag{25}$$

which is none other than the image under $\Gamma(\mathrm{Spec}(R)\backslash V((f_{i\in I})), \bullet)$ of the Cech complex with ordering of $\mathcal{O}_{\mathrm{Spec}(R)\backslash V((f_{i\in I}))}$ for the covering (D_{f_i}) of $\mathrm{Spec}(R)\backslash V((f_{i\in I}))$.

We shall write $C^\bullet_{\mathrm{aug}}((f_{i\in I}), R)$ for the *augmented Cech complex*

$$C^\bullet_{\mathrm{aug}}((f_{i\in I}), R) : 0 \to R \to C^0((f_{i\in I}), R) \stackrel{d^0}{\to} C^1((f_{i\in I}), R) \stackrel{d^1}{\to} \ldots$$

where R sits in degree 0.

Notice the following interesting fact. There is a canonical isomorphism of complexes

$$\otimes_{i\in I} C^\bullet_{\mathrm{aug}}(f_i, R) \simeq C^\bullet_{\mathrm{aug}}((f_{i\in I}), R) \tag{26}$$

Here by $C^\bullet_{\mathrm{aug}}(f_i, R)$ we mean the augmented Cech complex associated with the family with one element f_i. This can be checked by looking at the definitions of the differentials and the objects.

Complement 3.11. Suppose for the time of this complement that R is an $\mathbb{Z}$-graded ring and that the f_i are homogenous for the grading. Looking at the definitions of the maps, we see that the augmented Cech complex is a complex of graded modules and that the isomorphism (26) respects the grading.

For notational simplicity, write from now on $n = \#I$ and identify I with $\{1, \ldots, n\}$ as ordered sets.

We now suppose that $R = \mathbb{Z}[X_1, \ldots, X_n]$ and that $f_i = X_i$.

Lemma 3.12. *We have $\mathcal{H}^k(C^\bullet_{\mathrm{aug}}((f_{i\in I}), R)) = 0$ for all $k \neq \#I$.*

Proof. We shall prove this by induction on $\#I$.

The complex $C^\bullet_{\mathrm{aug}}((X_{i\in I}), R)$ is exact at 0, since R is a domain, so the statement holds for $\#I = 1$.

In particular, for any index i, the complex $C^\bullet_{\mathrm{aug}}(X_i, R)$ is quasi-isomorphic to $\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_i, R))[-1]$, since $C^\bullet_{\mathrm{aug}}(X_i, R)$ is a two term complex. Since $C^\bullet_{\mathrm{aug}}((X_{i\in I}), R)$ is also a complex of flat R-modules, Lemma 3.10

implies that

$$\begin{aligned}
& \mathcal{H}^k(C^\bullet_{\mathrm{aug}}((X_{i\in I}),R)) \\
= \ & \mathcal{H}^k(C^\bullet_{\mathrm{aug}}(X_1,R)\otimes C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R)) \simeq \mathcal{H}^k(\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1,R))[-1]\otimes C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R)) \\
\simeq \ & \mathcal{H}^{k-1}(\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1,R))\otimes C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R)) \\
\simeq \ & \mathrm{Tor}_{\#I-k}(\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1,R)),\mathcal{H}^{\#I-1}(C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R))).
\end{aligned}$$

where we used the inductive hypothesis. We shall now compute

$$\mathrm{Tor}_{\#I-k}(\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1,R)),\mathcal{H}^{\#I-1}(C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R))).$$

From the definitions, we see that

$$\mathcal{H}^{\#I-1}(C^\bullet_{\mathrm{aug}}((X_{i\in I,\ i\neq 1}),R)) = R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}})$$

and

$$\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1,R))\simeq R/R_{X_1}$$

Now consider the exact sequence of R-modules

$$0\to R\to R_{X_1}\to R/R_{X_1}\to 0$$

We consider the long exact cohomology sequence obtained when applying the right-exact functor

$$(\bullet)\otimes_R R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}})$$

to this sequence. Since R and R_{X_1} are flat R-modules, we obtain the sequence

$$\begin{aligned}
0 \ \to \ & \mathrm{Tor}_1(R/R_{X_1}, R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}}))\to R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}}) \\
\stackrel{(*)}{\to} \ & R_{X_1}\otimes_R R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}}) \\
\to \ & (R/R_{X_1})\otimes_R R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}})\to 0
\end{aligned}$$

and also that

$$\mathrm{Tor}_l(R/R_{X_1}, R_{X_2\cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}}))=0$$

for $l>1$. We shall now show that the map $(*)$ is injective. For this, it is sufficient to show that if

$$e\in R_{X_2\ldots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}})$$

and $e\cdot X_1^k=0$ for some $k\geqslant 0$ then $e=0$ (see [AM69, Prop. 3.5]). Let

$$\widetilde{e}\in R_{X_2\ldots X_n}$$

be a representative of e. If $e\cdot X_1^k=0$ in $R_{X_2\ldots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}})$, we have

$$\widetilde{e}\cdot X_1^k\in \sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1}\cdots X_{i_{n-2}}} \tag{27}$$

Now every element of $R_{X_2 \cdots X_n}$ has a unique expression, up to ordering, as a sum of monomials of the form $X_1^{l_1} \cdots X_n^{l_n}$, where $l_1 \geqslant 0$ and $l_i \in \mathbb{Z}$ for $i > 1$. On the other hand, an element of $R_{X_2 \cdots X_n}$ is in $\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1} \cdots X_{i_{n-2}}}$ if and only if it has an expression as a sum of monomials of the form $X_1^{l_1} \cdots X_n^{l_n}$, where $l_1 \geqslant 0$, $l_i \in \mathbb{Z}$ for $i > 1$, and for at least one $i_0 > 1$, we have $l_{i_0} \geqslant 0$. This condition is satisfied for $\widetilde{e}$ if and only if it is satisfied for $\widetilde{e} \cdot X_1^k$ and thus we conclude that $\widetilde{e} \in \sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1} \cdots X_{i_{n-2}}}$, which is what we wanted to show.

From the injectivity of $(*)$, we deduce that

$$\mathrm{Tor}_1(R/R_{X_1}, R_{X_2 \cdots X_n}/(\sum_{1<i_1<i_2<\cdots<i_{n-2}} R_{X_{i_1} \cdots X_{i_{n-2}}})) = 0$$

and thus the expression

$$\mathrm{Tor}_{\#I-k}(\mathcal{H}^1(C^\bullet_{\mathrm{aug}}(X_1, R)), \mathcal{H}^{\#I-1}(C^\bullet_{\mathrm{aug}}((X_{i \in I,\ i \neq 1}), R)))$$

vanishes if $k \neq \#I$, which is what the lemma asserts. □

Complement 3.13. Return to the situation of a general ring R. Suppose that R is a noetherian $\mathbb{N}$-graded ring. Suppose that the the f_i are homogenous elements of R. Suppose also that f_1 is not a zero-divisor of R, that the image of f_2 is not a zero divisor in $R/(f_1)$, that the image of f_3 is not a zero divisor in $R/(f_1, f_2)$ etc. Then Lemma 3.12 holds also.

Corollary 3.14. *We have* $H^k(\mathbb{A}^r \backslash \{0\}) = 0$ *if* $k \neq r - 1$.

The cohomology of projective space.

If $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^r}(1)$ is the canonical bundle on $\mathbb{P}^r$ and $n \geqslant 0$, we shall write $\mathcal{O}(n)$ for $\mathcal{O}(1)^{\otimes n}$. For $n < 0$, we also write

$$\mathcal{O}(n) := \mathcal{H}om(\mathcal{O}(-n), \mathcal{O}_{\mathbb{P}^r}) =: (\mathcal{O}(-n))^\vee$$

In general, if F is a locally free sheaf on a scheme X (see Exercise 3.7 for this notion), we write

$$F^\vee := \mathcal{H}om(F, \mathcal{O}_X)$$

Suppose that L is a locally free sheaf of rank 1 on an integral scheme X. If $\sigma \in L(X)$ and $\sigma \neq 0$, then σ induces a morphism of sheaves

$$\sigma^\vee : L^\vee \to \mathcal{O}_X$$

whose image is a quasi-coherent sheaf of ideals $\mathcal{I}$ by Corollary 2.20. Now by Exercise 3.10, the morphism $\sigma^\vee$ is a monomorphism and hence identifies $\mathcal{I}$ with $L^\vee$. If we let $\iota : Z(\sigma) \to X$ be the closed subscheme associated with $\mathcal{I}$ by Lemma 2.25, we thus have an exact sequence

$$0 \to L^\vee \to \mathcal{O}_X \to \iota_*(\mathcal{O}_{Z(\sigma)}) \to 0$$

We leave it to the reader to verify the following simple fact. If $f : X_0 \to X$ is a morphism of schemes, then there is a morphism $g : X_0 \to Z(\sigma)$ such that $f = \iota \circ g$ iff $f^*(\sigma^\vee) = 0$. Furthermore, the morphism g, if it exists, is then unique. From this we deduce $Z(\sigma)$ represents the functor **Schemes** $\mapsto$ **Sets**

$$S \mapsto \{f \in X(S) \,|\, f^*(\sigma^\vee) = 0\}$$

The closed subscheme $Z(\sigma)$ is called the *zero-scheme* associated with σ.

Lemma 3.15. *Let $f : X \to Y$ be an affine morphism of schemes. Suppose that X is noetherian. Then for all quasi-coherent sheaves F on X, we have $R^k f_*(F) = 0$ for all $k > 0$.*

Proof. By Proposition 1.14, we may suppose that Y (and thus X) is affine. Now the lemma follows from the fact that $f_* : \mathfrak{Q}coh(X) \to \mathfrak{Q}coh(Y)$ is an exact functor (by Lemma (2.22)), from the fact that injective $\mathcal{O}_X(X)$-modules are flasque (follows from Lemma 2.24) and from Complement 1.23. □

Lemma 3.16. *Suppose that X is noetherian and let $\iota : X \to Y$ be a closed immersion. Then ι is an affine morphism.*

Proof. Notice first that $\iota_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ is an exact functor, because if $x \in X$, we have natural functorial identifications of stalks $\iota_*(F)_{\iota(x)} = F_x$.

To show that ι is affine, we may suppose that Y is affine. Let F be a quasi-coherent sheaf on X and let $I^\bullet$ be a flasque quasi-coherent resolution of F. Then $\iota_*(I^\bullet)$ is a cochain complex of flasque quasi-coherent sheaves (use Proposition 2.32 and Lemma 2.24) and it is a resolution of $\iota_*(F)$, since ι_* is exact by the above remark. On the other hand, since Y is affine, we have

$$\mathcal{H}^k(\Gamma(Y, \iota_*(I^\bullet))) = \mathcal{H}^k(\Gamma(X, I^\bullet)) = H^k(X, F) = 0$$

for all $k \neq 0$ by Proposition 2.35. We conclude by appealing to Theorem 2.42. □

Complement 3.17. Let X be a noetherian scheme and suppose that X a finite open covering (U_i) such that any finite intersection of the U_i is affine. Let F be a quasi-coherent sheaf on X. Then we have canonically

$$\mathcal{H}^k(\Gamma(X, \underline{C}^\bullet((U_i), F))) \simeq H^k(X, F)$$

This follows from the existence of the Leray spectral sequence (see Corollary 1.24 and Complement 2.33) and from Lemma 3.15.

Proposition 3.18. *Let A be a noetherian ring. Then for all $n, k \in \mathbb{Z}$, $H^k(\mathbb{P}^r_A, \mathcal{O}(n))$ is a finitely generated A-module. Furthermore, we have*

$$H^0(\mathbb{P}^r_A, \mathcal{O}) \simeq A$$

and

$$H^k(\mathbb{P}^r_A, \mathcal{O}(n)) = 0$$

for all $k \geqslant 1$ and all $n \geqslant 0$.

Proof. First note that for any of the canonical sections $X_i = X_i \otimes_A 1 \in \Gamma(\mathbb{P}^r_A, \mathcal{O}(1))$, the zero scheme $Z(X_{i\in I})$ is canonically isomorphic to $\mathbb{P}^{r-1}_A$. Indeed, for any scheme over $\mathrm{Spec}(A)$ we have

$$\mathbb{P}^r_A(S) = \{\text{isomorphism classes of surjective morphisms } \phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}\}$$

and the $Z(X_{i\in I})$ thus represents the functor $\mathbf{Schemes}/\mathrm{Spec}(A) \mapsto \mathbf{Sets}$

$$S \mapsto \{\text{isomorphism classes of surjective morphisms } \phi = \oplus_k \phi_k : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L} \text{ such that } \phi_i = 0\}$$

which is isomorphic to the functor $\mathbb{P}^{r-1}_A(\bullet)$.

Thus we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}) \to 0$$

and tensoring this sequence with $\mathcal{O}(k)$ ($k \in \mathbb{Z}$), we obtain a sequence

$$0 \to \mathcal{O}(k-1) \to \mathcal{O}(k) \to \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) \to 0 \tag{28}$$

Note that $\iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) \simeq \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}) \otimes \mathcal{O}(k)$. This follows from Exercise 3.1 (which is easy to prove in the special case of a closed immersion). Now consider the long exact sequence associated with the sequence (28):

$$\begin{aligned} 0 \quad &\to \quad H^0(\mathbb{P}^r_A, \mathcal{O}(k-1)) \to H^0(\mathbb{P}^r_A, \mathcal{O}(k)) \to H^0(\mathbb{P}^{r-1}_A, \iota_*(\mathcal{O}(k))) \\ &\to \quad H^1(\mathbb{P}^r_A, \mathcal{O}(k-1)) \to H^1(\mathbb{P}^r_A, \mathcal{O}(k)) \to H^1(\mathbb{P}^{r-1}_A, \iota_*(\mathcal{O}(k))) \to \dots \end{aligned}$$

Now remember that closed immersions are affine by Lemma 3.16 and thus by Lemma 3.15 and Corollary 1.24 (or by a direct reasoning), we have canonically

$$H^i(\mathbb{P}^r_A, \iota_*(\mathcal{O}(k))) \simeq H^i(\mathbb{P}^{r-1}_A, \mathcal{O}(k)))$$

Thus by a double induction on r and k, we see that it is sufficient to prove that $H^0(\mathbb{P}^r_A, \mathcal{O}) = A$ and $H^k(\mathbb{P}^r_A, \mathcal{O}) = 0$ for all $k > 0$. This is what we shall now do.

We apply Complement 3.11 to the ring $R = A[X_0, \dots X_r]$ with its natural grading and to the family of the $f_i = X_i$. We obtain a graded complex

$$0 \to R \to \bigoplus_{i_0} R_{X_{i_0}} \to \bigoplus_{i_0 < i_1} R_{X_{i_0} X_{i_1}} \to \dots$$

and by Lemma 3.12 this complex is exact in degrees $\neq r+1$. If we take the part of homogenous degree 0 of this resolution, we get a sequence

$$0 \to A \to \bigoplus_{i_0} A[\frac{X_0}{X_{i_0}}, \dots, \frac{X_r}{X_{i_0}}] \to \bigoplus_{i_0 < i_1} A[\frac{X_0}{X_{i_0}}, \dots, \frac{X_r}{X_{i_0}}]_{\frac{X_{i_1}}{X_{i_0}}} \to \dots \tag{29}$$

The sequence (29) is by construction the image under $\Gamma(\mathbb{P}^r_A, \bullet)$ of the Cech complex for $\mathcal{O}_{\mathbb{P}^r_A}$ associated with the standard covering of $\mathbb{P}^r_A$ (see the proof of Theorem 3.4). Now the objects of the Cech complex are all $\Gamma(\mathbb{P}^r_A, \bullet)$-acyclic sheaves, because all the intersections of the open sets in the standard open covering are affine (use Lemma 3.16, Corollary 1.24 and Proposition 2.35). We have thus shown that $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) \simeq A$ and that $H^k(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) = 0$ for all $k \neq d$.

We shall now compute $H^d(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A})$. Examining the complex (29), we see that

$$H^d(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) \simeq A[X_0, \dots X_r]^{[0]}_{X_0 \dots X_r} / (\sum_{k=0}^{r} A[X_0, \dots, \widehat{X_k}, \dots X_r]^{[0]}_{X_0 \dots \widehat{X_k} \dots X_r})$$

The ring $A[X_0, \dots X_r]^{[0]}_{X_0 \dots X_r}$ is generated as an A-module by expressions of the form $\prod_{k=0}^r X_k^{l_k}$ where $l_k \in \mathbb{Z}$ and $\sum_{k=0}^r l_k = 0$. Now we have

$$\prod_{k=0}^{r} X_k^{l_k} \in \sum_{k=0}^{r} A[X_0, \dots, \widehat{X_k}, \dots X_r]^{[0]}_{X_0 \dots \widehat{X_k} \dots X_r} \tag{30}$$

if and only if there is $k_0 \in \{0, \dots, r\}$ such that $l_{k_0} \geqslant 0$. But there must be such a k_0 for otherwise we couldn't have $\sum_{k=0}^r l_k = 0$. Thus (30) always holds and we have $H^d(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) = 0$. □

Corollary 3.19. *The scheme $\mathbb{P}^r$ is not affine.*

Proof. If $\mathbb{P}^r$ were affine, then we would have $\mathbb{P}^r \simeq \mathrm{Spec}(\mathbb{Z})$, according to the theorem. □

3.5 Cohomological properties of strongly projective morphisms

A morphism of schemes $f : X \to S$ is called *strongly projective* (this is called projective by Hartshorne) if there is a factorisation $f = p \circ \iota$, where $\iota : X \to \mathbb{P}^r_S$ is a closed immersion and $p : \mathbb{P}^r_S \to S$ is the natural projection morphism.

Theorem 3.20. *Let $f : X \to S$ be a strongly projective morphism. Suppose that S is a noetherian scheme. Let F be a coherent sheaf on X. Then for all $k \geqslant 0$, the sheaf $R^k f_*(F)$ is coherent.*

Proof. We may assume that $S = \mathrm{Spec}(R)$, where R is a noetherian ring.

We first show the statement in the case where f is a closed immersion. In that case, since f is then affine (by Lemma 3.16), we may also assume that $X = \mathrm{Spec}(T)$ is affine. We then have $R^k f_*(F) = 0$ for all $k > 0$ by Lemma 3.15 and thus we only have to show that $f_*(F)$ is coherent. We know that $f_*(F)$ is quasi-coherent by Proposition 2.32 and thus we only have to show that $f_*(F)(S)$ is a finitely generated R-module. This is a consequence of Lemma 2.22.

By Complement 2.33, we may thus suppose that $X = \mathbb{P}^r_S$ and that f is the natural projection. Abusing language, we denote by $\mathcal{O}(1)$ the pull-back of the universal line bundle on $\mathbb{P}^r$ by the projection $\mathbb{P}^r_S \to \mathbb{P}^r$. Notice that $\mathcal{O}(1)$ is also an ample line bundle. This follows from Proposition 3.6 and Exercises 3.2, 3.3 and 3.4.

Let n_0 be such that $F \otimes \mathcal{O}(n_0) := F(n_0)$ is globally generated. Noticing that

$$\mathcal{O}(n_0) \otimes \mathcal{O}(-n_0) \simeq \mathcal{O}_{\mathbb{P}^r_S},$$

we obtain a surjection $\bigoplus_{k=0}^{f} \mathcal{O}(-n_0) \to F$ for some f. Denoting by K the kernel of this morphism, we get an exact sequence

$$0 \to K \to \bigoplus_{k=0}^{f} \mathcal{O}(-n_0) \to F \to 0 \tag{31}$$

Note that K is also a coherent sheaf, because $\mathbb{P}^r_S$ is a noetherian scheme. Notice also that we may compute to cohomology of F using the Cech complex with ordering associated with the standard open covering of $\mathbb{P}^r_S$. See Complement 3.17. The terms of this complex vanish in degrees $> r$. Thus we know that $R^k f_*(F) = 0$ for all $k > r$. Now looking at the long exact cohomology for f_* of (31) we obtain a surjection

$$R^r f_*(\bigoplus_{k=0}^{f} \mathcal{O}(-n_0)) \to R^r f_*(F)$$

Thus we see that $R^r f_*(F)$ is coherent, since $R^r f_*(\oplus_{k=0}^{f} \mathcal{O}(-n_0))$ is coherent by Proposition 3.18. Since F was arbitrary, we deduce that $R^r f_*(K)$ is also coherent. The long exact cohomology sequence again now shows that we have an exact sequence

$$R^{r-1} f_*(\bigoplus_{k=0}^{f} \mathcal{O}(-n_0)) \to R^{r-1} f_*(F) \to R^r f_*(K)$$

and thus $R^{r-1}f_*(F)$ is also coherent. Thus $R^{r-1}f_*(K)$ is also coherent and we may continue this way to show that $R^k f_*(F)$ is coherent for all k. □

Theorem 3.21 (Serre)**.** *Let $f : X \to \mathrm{Spec}(A)$ be a strongly projective morphism, where A is a noetherian ring. Let L be an ample line bundle. Let F be a coherent sheaf on X. Then there is $n_0 \geqslant 0$ such that $R^k f_*(F \otimes L^{\otimes n}) = 0$ for all $n \geqslant n_0$ and all $k > 0$.*

Proof. We may wrog replace L by one of its tensor powers. Hence, by Proposition 3.9, we may assume that there is a closed immersion $\iota : X \to \mathbb{P}^r_A$ over $\mathrm{Spec}(A)$ of X into some $\mathbb{P}^r_A$, such that $\iota^*(\mathcal{O}_{\mathbb{P}^r_A}(1)) = L$. As in the proof of Theorem 3.20, we may thus suppose that $X = \mathbb{P}^r_A$ for some $r \geqslant 0$. Remember also that $\mathbb{P}^r_A$ has a finite covering (U_i) (the standard covering) by open affine schemes, such that every intersection of the U_i is affine. Thus by Complement 7, there exists k_0 such that we have $H^k(\mathbb{P}^r_A, Q) = 0$ for all $k > k_0$. In fact, we may take $k_0 = r$. Let n_0 be sufficiently large so that $F \otimes L^{\otimes n_0}$ is generated by its global sections. In other words, we have an exact sequence

$$0 \to K \to \bigoplus_{i=1}^{r_0} \mathcal{O} \to F(n_0) \to 0 \tag{32}$$

Looking at the long exact cohomology sequence of (32), we get a surjection

$$H^{k_0}(\mathbb{P}^r_A, \bigoplus_{i=1}^{r_0} \mathcal{O}) \to H^{k_0}(\mathbb{P}^r_A, F(n_0)).$$

and thus $H^{k_0}(\mathbb{P}^r_A, F(n_0)) = 0$ by Proposition 3.18. Now take n_1 so that $K(n_1)$ is also globally generated. Then we also have $H^{k_0}(\mathbb{P}^r_A, K(n_1)) = 0$. Looking at the long exact cohomology sequence of the sequence

$$0 \to K(n_1) \to \bigoplus_{i=1}^{r_0} \mathcal{O}(n_1) \to F(n_0 + n_1) \to 0 \tag{33}$$

we get a surjection

$$H^{k_0 - 1}(\mathbb{P}^r_A, \bigoplus_{i=1}^{r_0} \mathcal{O}(n_1)) \to H^{k_0-1}(\mathbb{P}^r_A, F(n_0 + n_1))$$

and again by Proposition 3.18 we see that $H^{k_0-1}(\mathbb{P}^r_A, F(n_0 + n_1)) = 0$, unless $k_0 - 1 = 0$. Continuing this way, we conclude that $F(n)$ has no cohomology in positive degrees for n sufficiently large. □

3.6 Cohomological characterisation of ample line bundles

The following theorem is the converse of Theorem 3.21.

Theorem 3.22. *Let X be a noetherian scheme. Let L be a line bundle on X. Suppose that for all coherent sheaves F on X, there is $n_0 \geqslant 0$ such that $H^k(X, F \otimes L^{\otimes n}) = 0$ for all $n \geqslant n_0$ and all $k > 0$. Then L is ample.*

Proof. The proof is similar to the proof of Theorem 2.42.

So let P be a closed point in X. This exists by Exercise 2.15. Let U be an open affine neighbourhood of P such that $L|_U \simeq \mathcal{O}_U$ and let Y be the complement of U in X. We view P, Y and $P \cup Y$ as reduced closed subschemes of X, via Lemma 2.40. Let I_P, I_Y and $I_{P \cup Y}$ be the corresponding quasi-coherent sheaves of ideals. Note that we have canonically $\mathcal{O}_P(P) \simeq \kappa(P)$ and that this isomorphism describes the sheaf $\mathcal{O}_P$ entirely.

By construction, we have an exact sequence

$$0 \to I_{Y\cup P} \to I_Y \to \kappa(P) \to 0 \tag{34}$$

where $\kappa(P)$ denotes the direct image of $\mathcal{O}_P$ by the closed immersion $P \to X$. Now choose n_0 sufficiently large so that $H^1(X, I_{Y\cup P} \otimes L^{\otimes n_0}) = 0$. Applying Theorem 1.3 to the sequence (34) tensored by $L^{\otimes n_0}$ and to the functor $\Gamma(X, \bullet)$, we obtain an exact sequence

$$\Gamma(X, I_Y \otimes L^{\otimes n_0}) \to \Gamma(X, \kappa(P)) \to H^1(X, I_{Y\cup P} \otimes L^{\otimes n_0})$$

where we have identified non-canonically $\kappa(P)$ with $\kappa(P) \otimes L^{\otimes n_0}$. Since by assumption $H^1(X, I_{Y\cup P} \otimes L^{\otimes n_0}) = 0$, we get a surjection $\Gamma(X, I_Y \otimes L^{\otimes n_0}) \to \Gamma(X, \kappa(P))$. Let $f \in \Gamma(X, I_Y \otimes L^{\otimes n_0})$ be such that the image of f in $\Gamma(X, \kappa(P)) \simeq \kappa(P)$ is 1. We view f as an element of $\Gamma(X, L^{\otimes n_0})$ via the natural inclusion

$$\Gamma(X, I_Y \otimes L^{\otimes n_0}) \to \Gamma(X, L^{\otimes n_0}).$$

By construction, we have that $P \in X_f$ and also that $X_f \subseteq U$. In particular, X_f is affine, because it corresponds to a basic open set in U, since $L|_U$ is trivial. If $X \neq X_f$, we now repeat this reasoning for a closed point P_2 in $X\backslash X_f$ (this is possible because $X\backslash X_f$ is quasi-compact, since X is noetherian) and obtain an affine neighbourhood U_2 of P_2 and $f_2 \in \Gamma(X, \mathcal{O}_X)$ such that $P_2 \in X_{f_2}$ and X_{f_2} is affine and we repeat it for $P_3 \in X\backslash X_f \cup X_{f_2}$ etc. The sequence of the X_{f_i} must stop after a finite number of steps, and thus cover X, because X is a noetherian topological space (by Lemma 2.12).

We can thus exhibit a finite sequence $f = f_0 \in \Gamma(X, L^{\otimes n_0}), \dots, f_l \in \Gamma(X, L^{\otimes n_l})$ such that X_{f_i} is affine for all i and such that the X_{f_i} cover X. Replacing some of the f_i by tensor powers $f_i \otimes f_i \otimes \cdots \otimes f_i$ we may assume that all the n_i are equal. We can then conclude by Proposition 3.6. $\square$

3.7 Exercises

Exercise 3.1 (Projection formula)**.** *Let $f : X \to Y$ be morphism of schemes, where X is noetherian. Let F be a quasi-coherent sheaf on X and let M be a locally free sheaf (see Exercise 3.7 below) on Y. Prove that there is for all $k \geqslant 0$ a canonical isomorphism*

$$R^k f_*(F \otimes f^*(M)) \simeq R^k f_*(F) \otimes M$$

which is natural in F and M.

Exercise 3.2. *Let $f : X \to S$ be a morphism of schemes with the property (P), where (P) is one the following:*

- *affine;*
- *an open immersion;*
- *a closed immersion;*
- *locally of finite type;*
- *quasi-compact.*

Let $g : S' \to S$ be a morphism of schemes and let $f' : X_{S'} = X \times_S S' \to S'$ be the morphism obtained from f by base-change. Show that f' also has property (P). We say that property (P) is invariant under base-change.

Exercise 3.3. *Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes and let $f' : X_{S'} = X \times_S S' \to S'$ be the morphism obtained from f by base-change. Suppose that f is a closed immersion or that f is an open immersion. Prove that $\mathrm{Image}(f') = g^{-1}(\mathrm{Image}(f))$.*

Exercise 3.4. *Let M be line bundle on a scheme S and let $\sigma \in \Gamma(S, M)$. Let $g : S' \to S$ be a morphism of schemes. Show that $g^{-1}(S_\sigma) = S'_{g^*(\sigma)}$.*

Exercise 3.5. *Prove Lemma 3.7. [Hint: proceed as in the proof of Lemma 2.37.]*

Exercise 3.6. *Let $(T, \mathcal{O}_T)$ be a ringed space. Let F, G be $\mathcal{O}_T$-modules. Show that the presheaf on T, which associates with $U \in \mathrm{Top}(T)$ the abelian group $\mathrm{Mor}_{\mathcal{O}_U}(F|_U, G|_U)$ is a sheaf and that it has a natural structure of $\mathcal{O}_T$-module. This sheaf is denoted by $\mathcal{H}om(F, G)$.*

Exercise 3.7. *Let S be a scheme. Let F be a coherent sheaf on S. The sheaf F is called locally free if for every $s \in S$, there an open neighbourhood U of S, a natural number r and an isomorphism $F|_U \simeq \oplus_{k=1}^{r_U} \mathcal{O}_U$. Show that for a given s and U as above, the natural number r with the above property is unique, if it exists. Show that if F is locally free, then the number r depends on s only and that it is a locally constant function on S.*

Exercise 3.8. *Let $r \geqslant 2$. Prove that $\mathbb{A}^r_{\mathbb{C}} \backslash \{0\}$ is not affine.*

Exercise 3.9. *Show that there is a canonical isomorphism*

$$\Gamma(\mathbb{P}^r, \mathcal{O}(k)) \simeq \mathbb{Z}[X_0, \dots, X_r]^{[k]}$$

where $\mathbb{Z}[X_0, \dots, X_r]^{[k]}$ is the set of elements of $\mathbb{Z}[X_0, \dots, X_r]$, which are homogenous of degree k.

Exercise 3.10. *Let X be an integral scheme and let $\phi : F \to G$ be a morphism of coherent locally free sheaves on X. Let$U$ be an open affine subscheme of X. Suppose that $\phi(U) : F(U) \to G(U)$ is injective. Prove that ϕ is a monomorphism.*

Exercise 3.11. *Let X be a noetherian scheme and let X_{red} be the reduced closed subscheme of X associated with the closed subset X of X. Show that X_{red} is affine if and only if X is affine.*

4 Flat morphisms

Let $f : X \to Y$ be a morphism of schemes.

Definition 4.1. *Let F be a $\mathcal{O}_X$-module. We say that F is flat over Y at $x \in X$ if the stalk F_x is flat as a $\mathcal{O}_{Y,f(x)}$-module via the natural morphism of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. We say that F is flat over Y if F is flat at every $x \in X$.*

Recall that a module M over a ring R is flat if the functor $\bullet \otimes M$ from R-modules to R-modules sending an R-module N to $N \otimes M$ is an exact functor. See [AM69, Prop. 2.19].

Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism schemes and F a quasi-coherent sheaf on $\mathrm{Spec}(B)$. Let M be the B-module associated with F. Then F is flat over $\mathrm{Spec}(A)$ if and only if M is flat as an A-module: see Exercise 4.1.

Any base-change of a flat morphism is flat: Exercise 4.3.

We also recall without proof the following basic result:

Theorem 4.2. *Let A be a local ring and let M be a finite A-module. Then the following conditions on M are equivalent.*

- *M is flat over A;*
- *M is free over A.*

Proof. See Theorem 7.10 in *Commutative Ring Theory* by H. Matsumura (Cambridge University Press). □

A consequence of this theorem is that a coherent sheaf F on a noetherian scheme X is flat over X if and only if it is locally free (show this!).

4.1 Cohomology and flat base change

Theorem 4.3. *Let $f : X \to Y$ be a morphism of schemes. Suppose that Y is noetherian and affine and suppose that X has a finite open covering (U_i) of X such that any finite intersection of the U_i is affine. Let F be a quasi-coherent sheaf on X. Then for any cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{b} & Y \end{array}$$

where Y' is a noetherian and affine and b is flat, there is a canonical isomorphism of quasi-coherent sheaves

$$b^*(R^l f_*(F)) \simeq R^l f'_*(r^*(F)).$$

Proof. Consider the complex $L^\bullet := f_*(\underline{C}^\bullet((U_i), F))$. By construction we have

$$\mathcal{H}^l(b^*(L^\bullet)) \simeq R^l f'_*(r^*(F)).$$

Now since b is flat, we see that $\mathcal{H}^l(b^*(L^\bullet)) \simeq b^*(\mathcal{H}^l(L^\bullet))$, in other words we have $b^*(R^l f_*(F)) \simeq R^l f'_*(r^*(F))$. □

4.2 The semicontinuity theorem

Theorem 4.4. *Let $f : X \to Y$ be a morphism of schemes. Suppose that Y is noetherian and affine and suppose that X has a finite open covering (U_i) of X such that any finite intersection of the U_i is affine.*

Let F be a quasi-coherent sheaf on X and suppose that

- *the sheaf F is flat over Y;*
- *for all $l \geqslant 0$, the quasi-coherent sheaf $R^l f_*(F)$ is coherent.*

Then there is a finite cochain complex of coherent locally free modules $(K^\bullet)$ on Y with the following property. For any cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{b} & Y \end{array}$$

where Y' is a noetherian and affine, there is a canonical isomorphism of quasi-coherent sheaves

$$\mathcal{H}^l(b^*(K^\bullet)) \simeq R^l f'_*(r^*F)$$

Note that the assumptions of the theorem will be verified if f is a strongly projective morphism, F is flat over Y and Y is noetherian and affine.

Before, we begin with the proof, we shall prove some preliminary results in commutative algebra.

Lemma 4.5. *Let R be a noetherian ring. Let*

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

be a finite cochain complex of R-modules. Suppose that $\mathcal{H}^i(C^\bullet)$ is finitely generated for all $i \geqslant 0$. Then there is a finite cochain complex of R-modules

$$0 \to L^0 \to L^1 \to \cdots \to L^n \to 0$$

such that

- *$L^\bullet$ is quasi-isomorphic to $C^\bullet$;*
- *L^i is free for all $i > 0$;*
- *L^i is finitely generated for all $i \geqslant 0$.*

Furthermore, C^i is flat for all $i \geqslant 0$ then we may find a cochain complex $L^\bullet$ with the above properties such that L^0 is flat.

Proof. Suppose given a commutative diagram $(*)$

$$\begin{array}{ccccccccccccc} & & & & & & L^{m+1} & \xrightarrow{d_L^{m+1}} & L^{m+2} & \xrightarrow{d_L^{m+2}} & \cdots \longrightarrow & L^n & \longrightarrow 0 \\ & & & & & & \downarrow{\scriptstyle \phi_{m+1}} & & \downarrow{\scriptstyle \phi_{m+2}} & & & \downarrow{\scriptstyle \phi_n} & \\ 0 & \longrightarrow & C^0 & \xrightarrow{d_C^0} & \cdots & \longrightarrow & C^{m+1} & \xrightarrow{d_C^{m+1}} & C^{m+2} & \xrightarrow{d_C^{m+2}} & \cdots \longrightarrow & C^n & \longrightarrow 0 \end{array}$$

such that

- $L^{m+1} \to \cdots \to L^n$ is a cochain complex;
- ϕ_i induces an isomorphism $\mathcal{H}^i(L^\bullet) \simeq \mathcal{H}^i(C^\bullet)$ for all $i > m+1$;
- ϕ_{m+1} induces a surjection $\ker(d_L^{m+1}) \to \mathcal{H}^{m+1}(C^\bullet)$;
- for all $i \geqslant m+1$, L^i is free.

We shall show that we can extend this diagram to a diagram with the same properties with m in place of $m+1$. Let

$$B^{m+1} := \ker(\ker(d_L^{m+1}) \to \mathcal{H}^{m+1}(C^\bullet))$$

and let $\partial'_m : L'^m \to B^{m+1}$ be a surjection, where L'^m is a finitely generated free R-module. Note that B^{m+1} is finitely generated by the noetherian hypothesis. Similarly let L''^m be a finitely generated free R-module and let

$$L''^m \to \ker(d_C^m)$$

be a map of R-modules, inducing a surjection $L''^m \to \mathcal{H}^m(C^\bullet)$. Let

$$\phi''_m : L''^m \to C^m$$

be the induced map. Consider the diagram

$$\begin{array}{ccc} L'^m & \xrightarrow{\partial'_m} & L^{m+1} \\ & & \downarrow{\scriptstyle \phi_{m+1}} \\ C^m & \xrightarrow{d_C^m} & C^{m+1} \end{array}$$

and let $\phi'_m : L'^m \to C^m$ be any map making this diagram commutative. This makes sense, because by construction we have $\phi_{m+1} \circ \partial'_m \subseteq \mathrm{Image}(d_C^m)$.

Finally define

$$L^m := L'^m \oplus L''^m$$

and

$$\phi_m := \phi'_m \oplus \phi''_m \ , \ d_L^m := \partial'_m \oplus 0.$$

With these definitions, we have completed the extension of our diagram $(*)$ from $m+1$ to m. Since a diagram $(*)$ clearly exists for $m > n$ (just set $L^i = 0$ for all i), we see that by induction, the diagram $(*)$ exists for $m = -1$. Now replace L^0 by $L^0/(\ker(\phi_0) \cap \ker(d_L^0))$. The complex $L^\bullet$ now has all the requested properties.

We suppose now that all the C^i are flat. We wish to show that L^0 is also flat.

To see this, notice that for any R-module M, we have the two spectral sequences

$$E_{II}^{pq} = \mathrm{Tor}_{-p}(\mathcal{H}^q(L^\bullet), M) \Rightarrow \mathrm{Tor}_{-p-q}(L^\bullet, M)$$

and

$$E_I^{pq} = \mathrm{Tor}_{-q}(L^p, M) \Rightarrow \mathrm{Tor}_{-p-q}(L^\bullet, M)$$

and similarly for $C^\bullet$ (see Theorem 1.4). From this and the fact that these spectral sequences are functorial in $L^\bullet$, we conclude that

- $\mathrm{Tor}_{-r}(L^\bullet, M) \simeq \mathrm{Tor}_{-r}(C^\bullet, M)$ for all $r \in \mathbb{Z}$;
- $\mathrm{Tor}_{-r}(C^\bullet, M) = \mathcal{H}^r(C^\bullet \otimes M)$ for all $r \in \mathbb{Z}$;
- $\mathrm{Tor}_{-r}(L^\bullet, M) \simeq \mathrm{Tor}_{-r}(L^0, M)$ for all $r < 0$.

and thus that $\mathrm{Tor}_r(L^0, M) = 0$ for all $r > 0$. Since M was arbitrary, we conclude that L^0 is flat. $\square$

In the next lemmata, Nakayama's lemma will play a big role. We recall one of its formulations:

Lemma 4.6 (Nakayama's lemma)**.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $b_1, \ldots, b_k \in M$ be pairwise distinct elements. Then the set $\{b_1, \ldots, b_k\}$ is a set of generators of M of minimal cardinality if and only if the image of $\{b_1, \ldots, b_k\}$ in $M/\mathfrak{m}M$ is a basis of $M/\mathfrak{m}M$ as a $R/\mathfrak{m}$-vector space.*

See [AM69, p. 21] for the proof.

If R is a ring and $L^\bullet$ is a cochain complex of R-modules. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. We denote by $L^\bullet_\mathfrak{p}$ the complex on $R_\mathfrak{p}$ obtained by localisation and we write $L^\bullet(\mathfrak{p})$ for the complex $L^\bullet_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, which is a complex of $\kappa(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vector spaces.

Lemma 4.7. *Let R be a noetherian ring. Let M be a finitely generated R-module. Then the function $\dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p}))$ is upper semicontinuous on $\mathrm{Spec}(R)$, ie for all $n \in \mathbb{Z}$, the set*

$$\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p})) \geqslant n\}$$

is closed. If R is reduced and $\dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p}))$ is constant then M is locally free.

Proof. Let

$$R^t \to R^s \to M \to 0$$

be an exact sequence. Such a sequence exists because M is finitely generated and because of the noetherian hypothesis. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $(\phi_{ij})_{1 \leqslant i \leqslant s; 1 \leqslant i \leqslant t}$ be a $s \times t$ matrix representing the map $R^t \to R^s$ in the standard bases. For each $l \geqslant 1$, let $f_{1l}, \ldots f_{k_l l}$ be the set of all the minors of order l of (ϕ_{ij}) (these are polynomials in the ϕ_{ij}). We then have

$$\begin{aligned} & \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p})) \geqslant n\} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid s - \mathrm{rk}((\phi_{ij}(\mathfrak{p}))) \geqslant n\} \\ = \; & \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{rk}((\phi_{ij}(\mathfrak{p}))) \leqslant s - n\} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \forall l > s-n,\ r \geqslant 1 : f_{lr} \in \mathfrak{p}\} \\ = \; & \cap_{l > s-n} \cap_{r=1}^{k_l} V((f_{lr})) \end{aligned}$$

proving the first assertion in the lemma.

For the second assertion, let $\mathfrak{p} \in \mathrm{Spec}(R)$ and let $\gamma_1, \ldots, \gamma_r$ be a basis of $M(\mathfrak{p})$. We have to show that $\widetilde{M}$ is locally free in a neighborhood of $\mathfrak{p}$. Lift this basis to a set $a_1/b_1, \ldots, a_r/b_r \in M_\mathfrak{p}$, where $b_1, \ldots b_r \in R \backslash \mathfrak{p}$. We may and do replace R by $R_{b_1 \cdots b_r}$, since $R_{b_1 \cdots b_r}$ corresponds to a basic open set of R. Consider now the exact sequence of R-modules

$$0 \to K \to R^r \xrightarrow{\phi} M \to C \to 0 \tag{35}$$

where $\phi((x_1, \ldots, x_r)) = \sum_i x_i \cdot \frac{a_i}{b_i}$. Now by construction $C(\mathfrak{p}) = 0$ and by Nakayama's lemma, we conclude that $C_\mathfrak{p} = 0$. Since C is a finitely generated R-module, this means that there exists $b \in R \backslash \mathfrak{p}$ such that $b \cdot C = 0$ and thus replacing again R by R_b, we obtain a sequence of R-modules

$$0 \to K \to R^r \xrightarrow{\phi} M \to 0$$

Now K is a finitely generated R-module as well, since R is noetherian and we choose a surjection $R^t \to K$. This yields another exact sequence of R-modules

$$R^t \xrightarrow{\lambda} R^r \xrightarrow{\phi} M \to 0$$

Now by the assumption that $\dim_{\kappa(\mathfrak{q})}(M(\mathfrak{q})) = r$ for all $\mathfrak{q} \in \operatorname{Spec}(R)$, we see that for all $\mathfrak{q} \in \operatorname{Spec}(R)$, the map $\phi(\mathfrak{q})$ is an isomorphism and thus $\lambda(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \operatorname{Spec}(R)$. Now the map λ can be described by a matrix $(\psi_{ij} \in R)$ and we have just shown that for all i, j and all $\mathfrak{q} \in \operatorname{Spec}(R)$, we have $\psi_{ij}(\mathfrak{q}) = 0$. In other words, for any pair of indices i, j, the elements ψ_{ij} is in the nilradical of R. But the nilradical of R is 0 by assumption and thus $\lambda = 0$. We conclude that ϕ is an isomorphism and thus M is free. □

Lemma 4.8. *Let R be a reduced noetherian ring. Let*

$$0 \to L^0 \xrightarrow{d^0} L^1 \xrightarrow{d^1} \cdots \to L^n \to 0$$

be a finite cochain complex of finitely generated free R-modules. Suppose that the function on $\operatorname{Spec}(R)$

$$\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p})))$$

is constant. Then $H^i(L^\bullet)$ is free and there is a natural isomorphism

$$H^i(L^\bullet)(\mathfrak{p}) \simeq H^i(L^\bullet(\mathfrak{p}))$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. Consider the following portion of the $L^\bullet$:

$$L^{i-1} \xrightarrow{d^{i-1}} L^i \xrightarrow{d^i} L^{i+1}.$$

By assumption, the function of $\mathfrak{p} \in \operatorname{Spec}(R)$ given by

$$\operatorname{nullity}(d^i(\mathfrak{p})) - \operatorname{rk}(d^{i-1}(\mathfrak{p}))$$

is constant. By the rank nullity formula, we now that

$$\operatorname{nullity}(d^i(\mathfrak{p})) = \dim_R(L^i) - \operatorname{rk}(d^i(\mathfrak{p}))$$

and thus the quantity $\operatorname{rk}(d^i(\mathfrak{p})) + \operatorname{rk}(d^{i-1}(\mathfrak{p}))$ is constant on $\operatorname{Spec}(R)$. Remember that by Lemma 4.7, the quantity $-\operatorname{rk}(d^i(\mathfrak{p}))$ is upper semicontinuous. On the other hand, since $\operatorname{rk}(d^i(\mathfrak{p})) + \operatorname{rk}(d^{i-1}(\mathfrak{p}))$ is constant, the quantity $\operatorname{rk}(d^i(\mathfrak{p}))$ is also upper semicontinuous. Hence $\operatorname{rk}(d^i(\mathfrak{p}))$ is continuous, or in other words locally constant. Similarly $\operatorname{rk}(d^{i-1}(\mathfrak{p}))$ is locally constant.

Now consider the sequence

$$L^{i-1} \xrightarrow{d^{i-1}} L^i \to L^i/L^{i-1} \to 0$$

where $L^i/L^{i-1} := L^i/\operatorname{Image}(d^{i-1})$. Reducing $\bmod\ \mathfrak{p}$, we obtain a sequence

$$L^{i-1}(\mathfrak{p}) \xrightarrow{d^{i-1}(\mathfrak{p})} L^i(\mathfrak{p}) \to (L^i/L^{i-1})(\mathfrak{p}) \to 0$$

and since $\operatorname{rk}(d^{i-1}(\mathfrak{p}))$ is locally constant, we see that $\operatorname{rk}((L^i/L^{i-1})(\mathfrak{p}))$ is also locally constant and we conclude from Lemma 4.7 that L^i/L^{i-1} is locally free. Similarly, L^{i+1}/L^i is locally free.

Lastly, consider the sequence

$$0 \to \mathcal{H}^i(L^\bullet) \to L^i/L^{i-1} \to L^{i+1} \to L^{i+1}/L^i \to 0$$

where the differentials are the obvious ones. This sequence is by construction exact. Since L^i/L^{i-1}, L^{i+1} and L^{i+1}/L^i are locally free, we see that $\mathcal{H}^i(L^\bullet)$ is also locally free (show this!) and that the sequence

$$0 \to \mathcal{H}^i(L^\bullet)(\mathfrak{p}) \to (L^i/L^{i-1})(\mathfrak{p}) \to L^{i+1}(\mathfrak{p}) \to (L^{i+1}/L^i)(\mathfrak{p}) \to 0$$

is also exact. Now the analogous sequence

$$0 \to \mathcal{H}^i(L^\bullet(\mathfrak{p})) \to L^i(\mathfrak{p})/L^{i-1}(\mathfrak{p}) \to L^{i+1}(\mathfrak{p}) \to L^{i+1}(\mathfrak{p})/L^i(\mathfrak{p}) \to 0$$

is also exact and we have

$$L^i(\mathfrak{p})/L^{i-1}(\mathfrak{p}) = (L^i/L^{i-1})(\mathfrak{p})$$

and

$$L^{i+1}(\mathfrak{p})/L^i(\mathfrak{p}) = (L^{i+1}/L^i)(\mathfrak{p}).$$

This gives a canonical isomorphism $\mathcal{H}^i(L^\bullet)(\mathfrak{p}) \simeq \mathcal{H}^i(L^\bullet(\mathfrak{p}))$. □

Lemma 4.9. *Let R be a noetherian ring. Let*

$$0 \to L^0 \to L^1 \to \cdots \to L^n \to 0$$

be a finite cochain complex of finitely generated free R-modules. Then the function on $\mathrm{Spec}(R)$

$$\mathfrak{p} \mapsto \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p})))$$

is locally constant on R.

Proof. Notice that

$$\sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p}))) = \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(L^\bullet(\mathfrak{p})) = \sum_{i \geqslant 0} (-1)^i \mathrm{rk}(L^\bullet)$$

We leave it to the reader as an exercise to check this (hint: lift bases). The lemma follows from this. □

We can now turn to the proof of the semicontinuity theorem.

Proof. (of Theorem 4.4) Consider the complex $L^\bullet := f_*(\underline{C}^\bullet((U_i), F))$. Then L^i are flat and quasi-coherent, $L^\bullet$ is a finite complex and by construction for any cartesian diagram as in the statement of Theorem 4.4, we have

$$\mathcal{H}^l(b^*(L^\bullet)) \simeq R^l f'_*(r^* F).$$

Hence we only have to show that there exists a complex of coherent locally free modules $K^\bullet$, which is quasi-isomorphic to $L^\bullet$. This complex will have all the required properties by Lemma 3.10. To conclude, apply Lemma 4.5. □

Corollary 4.10. *Let $f : X \to Y$ be a strongly projective morphism. Suppose that Y is noetherian. Let F be a coherent sheaf on X and suppose that F is flat over Y. Then the function*

$$y \mapsto \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(y)}(H^i(X_y, F_y))$$

is locally constant on Y.

Proof. Apply Lemma 4.9 to the complex $K^\bullet$ provided by the semicontinuity theorem. □

Corollary 4.11. *Let $f : X \to Y$ be a strongly projective morphism. Suppose that Y is noetherian and reduced. Let F be a coherent sheaf on X and suppose that F is flat over Y. Suppose that the function*

$$y \mapsto \dim_{\kappa(y)}(H^i(X_y, F_y))$$

is locally constant on Y. Then $R^i f_*(F)$ is locally free.

Proof. Apply Lemma 4.8 to the complex $K^\bullet$ provided by the semicontinuity theorem. □

4.3 Hilbert polynomials

Let $r \geqslant 0$ and let K be a field. Let F be a coherent sheaf on $\mathbb{P}^r_K$. For all $n \in \mathbb{Z}$, we write

$$\chi_F(n) := \sum_{i \geqslant 0} (-1)^i \dim_K H^i(X, F \otimes \mathcal{O}(n))$$

Proposition 4.12. *The function $\chi_F(\bullet)$ is a polynomial with rational coefficients.*

The polynomial $\chi_F(\bullet)$ is called the *Hilbert polynomial* of F.

Proof. If G is a quasi-coherent sheaf on a locally noetherian scheme, we shall write $\mathrm{SS}(G)$ for the closed subscheme associated with the annihilator $\mathrm{Ann}(G)$ (see Exercise 4.5).

Notice first that the statement clearly holds if F is the zero sheaf.

By noetherian induction (see 4.8), we may thus suppose that the Proposition holds if $\mathrm{SS}(F) \neq X$.

Now consider the sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to Z(X_0) \to 0$$

associated with the section X_0 of $\mathcal{O}(1)$. Tensoring this sequence with F, we obtain a sequence

$$L^\bullet : 0 \to K \to F(-1) \to F \to F \otimes Z(X_0) \to 0 \tag{36}$$

where K sits in degree 0. If we consider the spectral sequence

$$E_2^{pq} = H^q(\mathbb{P}^r_K, L^p) \Rightarrow H^{p+q}(\mathbb{P}^r_K, L^\bullet)$$

we see that

$$\sum_{p,q} (-1)^{p+q} \dim_K(H^q(\mathbb{P}^r_K, L^p)) = \sum_k (-1)^k \dim_K(H^k(\mathbb{P}^r_K, L^\bullet))$$

Now notice that

$$\sum_{p,q} (-1)^{p+q} \dim_K(H^q(\mathbb{P}^r_K, L^p)) = \sum_k (-1)^k \chi_{L^k}(0)$$

and that $H^k(\mathbb{P}^r_K, L^\bullet) = 0$ for all k, since the sequence (36) is exact. Making the same computation for the sequence (36) tensored with $\mathcal{O}(n)$, we conclude that

$$\sum_k (-1)^k \chi_{L^k}(\bullet) = 0$$

The same reasoning clearly applies for any bounded sequence of coherent sheaves on $\mathbb{P}^r_K$ in lieu of $L^\bullet$.

Now notice that

$$K|_{\mathbb{P}^r_{K,X_0}} = F \otimes Z(X_0)|_{\mathbb{P}^r_{K,X_0}} = 0$$

and so $\mathrm{SS}(K) \neq X$ and $\mathrm{SS}(F \otimes Z(X_0)) \neq X$. By noetherian induction, we see that the function $\chi_F(n) - \chi_F(n-1)$ is a polynomial in n. Let $P(n) := \chi_F(n) - \chi_F(n-1)$. We have

$$\chi_F(n) = \chi_F(0) + \sum_{k=1}^{n} P(k)$$

Now notice that for all $i \geqslant 0$, the function $\sum_{k=1}^n k^i$ is a polynomial in n with rational coefficients (we have $\sum_{k=1}^n k^0 = n$, $\sum_{k=1}^n k = n(n+1)/2$ etc.). Thus $\chi_F(n)$ is a polynomial in n with rational coefficients. □

Complement 4.13. We record the following fact, which was established in the proof of Proposition 4.12. If

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of coherent sheaves on $\mathbb{P}^r_K$, then we have

$$\chi_F(n) = \chi_{F'}(n) + \chi_{F''}(n)$$

for all $n \in \mathbb{Z}$.

Example 4.1. We have

$$\chi_{\mathcal{O}_{\mathbb{P}^r_K}}(n) = \binom{n+r}{n}$$

We leave the verification of this formula as an exercise for the reader (hint: use Exercise 3.9).

Proposition 4.14. *Let S be a connected locally noetherian scheme, let $r \geqslant 0$ and let $\iota : X \to \mathbb{P}^r_S$ be a closed subscheme of $\mathbb{P}^r_S$. Suppose that X is flat over* $\mathrm{Spec}(A)$*. Then the Hilbert polynomial of $\iota_{\kappa(s)} : X_{\kappa(s)} \to \mathbb{P}^r_{\kappa(s)}$ does not depend on $\mathfrak{p} \in S$.*

Here the immersion $\iota_{\kappa(s)} : X_{\kappa(s)} \to \mathbb{P}^r_{\kappa(s)}$ is obtained by base-change from $\iota : X \to \mathbb{P}^r_S$ via the natural morphism $\mathrm{Spec}(\kappa(s)) \to S$.

Proof. Follows from Corollary 4.10. □

4.4 exercises

Exercise 4.1. *Let* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *be a morphism schemes and F a quasi-coherent sheaf on* $\mathrm{Spec}(B)$*. Let M be the B-module associated with F. Show that F is flat over* $\mathrm{Spec}(A)$ *if and only if M is flat as an A-module.*

Exercise 4.2. *A map of sets $S \to T$ is quasi-finite if for all $t \in T$, the set $f^{-1}(t)$ is finite. Let $f : X \to Y$ be an affine and strongly projective morphism of schemes. Assume that Y is noetherian. Show that f is quasi-finite.*

Exercise 4.3. *Let $f : X \to Y$ be a morphism of schemes and let F be a quasi-coherent sheaf on X. Suppose that F is flat over Y. Let*

$$\begin{array}{ccc} X' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{b} & Y \end{array}$$

be a cartesian diagram. Show that $r^*(F)$ is flat over Y'.

Exercise 4.4. *Let X be a locally noetherian scheme. Let F, G be quasi-coherent sheaves on X. Suppose that F is coherent. Show that the $\mathcal{O}_X$-module $\mathcal{H}om(F, G)$ is quasi-coherent. Show that $\mathcal{H}om(F, G)$ is coherent if F and G are coherent.*

Exercise 4.5 (Annihilators). *Let R be a ring and let M be an R-module. Define*

$$\mathrm{Ann}(M) := \{r \in R \,|\, r \cdot M = 0\}$$

Show that $\mathrm{Ann}(M)$ is an ideal in R.

Let now F be a $\mathcal{O}_X$-module on a ringed space X. Show that multiplication by scalars induces an arrow

$$m : \mathcal{O}_X \to \mathcal{H}om(F, F)$$

Suppose now that X is a locally noetherian scheme and that F is quasi-coherent. Show that $\ker(m)$ is a coherent sheaf of ideals. Let U be an open affine subscheme of X. Show that the ideal of $\Gamma(U, \mathcal{O}_X)$ corresponding to $\ker(m)|_U$ is the annihilator of $\Gamma(U, F|_U)$. We write $\mathrm{Ann}(F) := \ker(m)$.

Exercise 4.6. *Let X be a locally noetherian scheme and let F be a quasi-coherent sheaf. Let $\iota : Z \hookrightarrow X$ be the closed subscheme of X associated with $\mathrm{Ann}(F)$. Show that the natural morphism $F \to \iota_*(\iota^*(F))$ is an isomorphism.*

Exercise 4.7 (noetherian induction for topological spaces). *Let T be a noetherian topological space. Let $P(\bullet)$ be a property of closed subsets of T. Suppose that P(empty set) holds and that for all closed subsets C of T, the statement*

if $P(C')$ holds for all closed subsets $C' \overset{\neq}{\hookrightarrow} C$ then $P(C)$ holds

is verified. Then $P(T)$ holds.

Exercise 4.8 (noetherian induction for schemes). *Let T be a noetherian scheme. Let $P(\bullet)$ be a property of closed subschemes of T. Suppose that P(empty scheme) holds and that for all closed subschemes C of T, the statement*

if $P(C')$ holds for all closed subschemes $C' \overset{\neq}{\hookrightarrow} C$ then $P(C)$ holds

is verified. Then $P(T)$ holds.

5 Further results on the Zariski topology

5.1 Irreducible components

Definition 5.1. *Let T be a topological space. We say that T is irreducible if every non empty open subset of T is dense in T.*

Equivalently, T is irreducible iff no proper closed subset of T contains an open subset of T or if there is no pair of disjoint non empty open subsets in T. Notice that every open subset of an irreducible topological space is irreducible.

Lemma 5.2. *If A is a ring then $\operatorname{Spec}(A)$ is irreducible if and only if $A/\sqrt{(0)}$ is an integral ring.*

Proof. Since irreducibility concerns $\operatorname{Spec}(A)$ only as a topological space, we may wrog replace A by $A/\sqrt{(0)}$, since the corresponding morphism $\operatorname{Spec}(A/\sqrt{(0)}) \to \operatorname{Spec}(A)$ is a homeomorphism (because it is a surjective closed immersion). Thus we may assume that $\sqrt{(0)} = (0)$ or in other words that A has no non-vanishing nilpotent elements.

Suppose first that $\operatorname{Spec}(A)$ is irreducible and suppose that A were not integral. Then there is $f, g \in A$, with $f, g \neq 0$ and $f \cdot g = 0$. The open subsets $D_f(A)$ and $D_g(A)$ are then disjoint and non-empty. To see this, suppose that for some $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $f \notin \mathfrak{p}$, ie $\mathfrak{p} \in D_f(A)$. Since $f \cdot g = 0 \in \mathfrak{p}$, we thus have $g \in \mathfrak{p}$, since $\mathfrak{p}$ is prime. Hence $\mathfrak{p} \notin D_g(A)$. Thus we have $D_f(A) \cap D_g(A) = \emptyset$. and thus $\operatorname{Spec}(A)$ is not irreducible, contradicting the assumption.

Now suppose that A is integral. If A is not irreducible, there are two non empty disjoint open subsets in $\operatorname{Spec}(A)$. Reducing their sizes, we may assume that they are basic open set $D_f(A)$ and $D_g(A)$ for some $f, g \in A$ with $f, g \neq 0$. From the disjointness, we conclude that for every $\mathfrak{p} \in \operatorname{Spec}(A)$, we have either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. In particular for every $\mathfrak{p} \in \operatorname{Spec}(A)$, we have $f \cdot g \in \mathfrak{p}$. This implies that $f \cdot g$ is in the nilradical of A, which is zero by assumption so $f \cdot g = 0$, a contradiction. So A must be irreducible. □

Corollary 5.3. *Let A be a ring. Let $I \subseteq A$ be an ideal. Then $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.*

Lemma 5.4. *Let T be a noetherian topological space. There is a finite sequence $C_1, \dots C_k$ of closed irreducible subsets of T such that*

- $\bigcup_i C_i = T$;
- *for all indices i, we have $C_i \not\subseteq \cup_{j \neq i} C_j$.*

This sequence is unique up to permutation of the indices.

Proof. See Exercise 5.1. □

Complement 5.5. If we apply Lemma 5.4 and Corollary 5.3 to $\operatorname{Spec}(A)$, where A is a noetherian ring, we obtain the following statement. There is a finite sequence $\mathfrak{p}_1, \dots \mathfrak{p}_k$ of prime ideals in A such that

- $\bigcap_i \mathfrak{p}_i = \sqrt{0}$;
- for all indices i, we have $\mathfrak{p}_i \not\supseteq \cap_{j \neq i} \mathfrak{p}_j$.

This sequence is unique up to permutation of the indices. These ideals are called the *minimal prime ideals* of A.

In particular, if I is an ideal in a noetherian ring A, there is a finite sequence $\mathfrak{p}_1, \dots \mathfrak{p}_k$ of prime ideals in A such that

- $\bigcap_i \mathfrak{p}_i = \sqrt{I}$;
- for all indices i, we have $\mathfrak{p}_i \not\supseteq \cap_{j \neq i} \mathfrak{p}_j$.

This sequence is unique up to permutation of the indices.

The following lemma points out a specific property of irreducible closed subsets of schemes.

Lemma 5.6 (generic points)**.** *Let S be a scheme. Let $C \subseteq S$ be an irreducible closed subset. There is a unique point $\eta \in C$ such that the Zariski closure $\bar{\eta}$ is C.*

The point η is called *the generic point* of C.

Proof. We may wrog replace S by the reduced closed subscheme of S obtained from C (see Remark 2.41). We thus have to show that if S is an irreducible scheme then it has a unique generic point η. Since this question concerns only the topology of S, we way also suppose that S is reduced.

Now suppose to begin with that $S = \mathrm{Spec}(A)$ is affine. Then Lemma 5.2 shows that A is a domain. Now $\mathfrak{p} \in \mathrm{Spec}(A)$ be a prime ideal. The Zariski closure of $\mathfrak{p}$ is a closed set $V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$ such that

- $\mathfrak{a}$ is a radical ideal;
- $\mathfrak{p} \in V(\mathfrak{a})$;
- if $\mathfrak{p} \in V(\mathfrak{c})$ for some radical ideal $\mathfrak{c}$ then $V(\mathfrak{c}) \supseteq V(\mathfrak{a})$.

Translated into the language of ideals, this gives:

- $\mathfrak{a}$ is a radical ideal;
- $\mathfrak{a} \subseteq \mathfrak{p}$;
- if $\mathfrak{c} \subseteq \mathfrak{p}$, where $\mathfrak{c}$ is a radical ideal, then $\mathfrak{c} \subseteq \mathfrak{a}$.

From this we deduce that $\mathfrak{a} = \mathfrak{p}$ or in other words that the Zariski closure of $\mathfrak{p} \in \mathrm{Spec}(A)$ is $V(\mathfrak{p})$. Now notice that $(0) \in \mathrm{Spec}(R)$ has the property that $V((0)) = \mathrm{Spec}(R)$ so $\mathrm{Spec}(R)$ has at least one generic point. On the other hand, if $V(\mathfrak{p}) = \mathrm{Spec}(R)$ for some prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\sqrt{\mathfrak{p}} = \mathfrak{p} = \sqrt{(0)} = (0)$ (since R is a domain) so that the generic point is unique. This proves the lemma when S is affine.

In general, let (U_i) be an open affine covering of S. Each U_i is reduced and irreducible and thus by the above there is a unique $\eta_i \in U_i$ such that $\bar{\eta}_i \cap U_i = U_i$. Let i, j be any pair of indices and let $V_k \subseteq U_i \cap U_j$ be a non-empty open affine subset (note that $U_i \cap U_j \neq \emptyset$ because S is irreducible). Notice that by construction $\eta_i, \eta_j \in V_k$ and $\bar{\eta}_i \cap V_k = \bar{\eta}_j \cap V_k = V_k$. On the other hand V_k also has a unique generic point by the above so $\eta_i = \eta_j$. Hence all the η_i are equal to one $\eta \in S$. We conclude that

$$\bigcup_i (\bar{\eta} \cap U_i) = \bar{\eta} \cap \bigcup_i U_i = S$$

and thus η is a generic point of S. To see that it is unique let η' be another generic point of S. Let $U \subseteq S$ be an open affine subscheme. Then $\eta, \eta' \in U$ and η and η' are also generic points of U. Hence $\eta = \eta'$ by the above. □

5.2 Constructibility

Definition 5.7. *Let T be a noetherian topological space. A subset $E \subseteq T$ is called constructible if E is a finite union of locally closed subsets.*

The class of constructible sets is the smallest subclass of the power set of T, which contains the open subsets of T and is closed under finite unions and complementation (prove this!).

5.3 Chevalley's theorem

Theorem 5.8 (Chevalley-Tarski). *Let $f : X \to Y$ be a morphism of finite type. Suppose that Y is noetherian. Let $E \subseteq X$ be a constructible subset of X. Then $f(E)$ is a constructible subset of Y.*

To prove Theorem 5.8, we shall need some preliminary results in commutative algebra.

Theorem 5.9 (Noether's normalisation lemma). *Let K be a field and let A be a finitely generated K-algebra. Then there is a natural number $n \in \mathbb{N}$ and a map of K-algebras*

$$\phi : K[T_1, \dots T_n] \to A$$

such that ϕ is injective and finite.

Note that the map ϕ endows A with a $K[T_1, \dots T_n]$-algebra structure. By definition, ϕ is *finite* if A is a finitely generated $K[T_1, \dots T_n]$-module.

Proof. See [AM69, p. 69]. □

Theorem 5.10 (Going up theorem; [AM69, Th. 5.11]). *Let $\phi : A \to B$ be a morphism of rings and suppose that ϕ is injective and finite. Then $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.*

To prove Theorem 5.10, we shall need the following lemmata.

Lemma 5.11. *Let k be a domain let M be a finite k-module. Let $\phi : M \to M$ be a map of k-modules. Then is $n \in \mathbb{N}$ and $P(t) = t^n + a_{n-1} \cdot t^{n-1} + \cdots + a_0$ with $a_i \in k$ such that $P(\phi) = 0$.*

Proof. (of lemma 5.11) Let $h \in \mathbb{N}$ and let $s : k^h \to M$ be a surjective map of k-modules. Let $\widetilde{\phi} : k^h \to k^h$ be any map of k-modules such that $s \circ \widetilde{\phi} = \phi \circ s$. Then there is $n \in \mathbb{N}$ and $P(t) = t^n + a_{n-1} \cdot t^{n-1} + \cdots + a_0$ with $a_i \in k$ such that $P(\widetilde{\phi}) = 0$ (by Cayley-Hamilton). Thus

$$P(\phi) \circ s = s \circ P(\widetilde{\phi}) = 0$$

and since s is surjective, we see that $P(\phi) = 0$. □

Complement 5.12. Lemma 5.11 is also true without the assumption that k is a domain. This follows from the generalised Cayley-Hamilton theorem. See Theorem 2.1 in *Commutative Algebra* by H. Matsumura for this.

Lemma 5.13. *Suppose that $\lambda : k \to B_0$ is an injective and finite map of domains. Then B_0 is a field if and only if k is a field.*

Proof. (of Lemma 5.13). Suppose that k is a field. By induction on the number of generators of B_0 as a k-module, we may suppose that B_0 is generated by one element $b_0 \in B_0$ over k. Let $k[t] \to B_0$ be the k-algebra map sending t on b_0. The kernel of this map is a prime ideal, since B_0 is integral. Since prime ideals in $k[t]$ are maximal, we conclude that B_0 is a field.

No suppose that B_0 is a field. We want to show that k is a field. Let $x \in k^*$. We only have to show that the inverse $x^{-1} \in B_0$ lies in k. Let $e_x : B_0 \to B_0$ be the map such that $e_x(z) = z/x$ for all $z \in B_0$. By Lemma 5.11, there is a polynomial $P(t) = t^n + a_{n-1} \cdot t^{n-1} + \cdots + a_0 \in k[t]$ such that $P(e_x) = 0$. In particular, we have $P(e_x)(1) = P(1/x) = 0$. Thus we have $x^{n-1} \cdot P(1/x) = 0$, ie

$$x^{-1} + a_{n-1}x + \cdots + a_0 \cdot x^{n-1} = 0$$

which implies that $x^{-1} \in k$. □

Proof. (of Theorem 5.10) Let $\mathfrak{p} \in \mathrm{Spec}(A)$. The localised map $\phi_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p}$ is also finite and injective. Here $B_\mathfrak{p}$ is the localisation of B viewed as an A-module. It is easily checked that $B_\mathfrak{p}$ is naturally isomorphic to the localisation $B_{\phi(A\backslash\mathfrak{p})}$ of the ring B at the multiplicative set $\phi(A\backslash\mathfrak{p})$ and that via this isomorphism the map $\phi_\mathfrak{p}$ is a map of rings. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B_\mathfrak{p}) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow{\scriptstyle \mathrm{Spec}(\phi_\mathfrak{p})} & & \downarrow{\scriptstyle \mathrm{Spec}(\phi)} \\ \mathrm{Spec}(A_\mathfrak{p}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Since $\mathfrak{p}$ is the image of the maximal ideal $\mathfrak{m}$ of $A_\mathfrak{p}$ under the map $\mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$, it is sufficient to show that there is a prime ideal $\mathfrak{q}$ in $B_\mathfrak{p}$ so that $\phi_\mathfrak{p}^{-1}(\mathfrak{q}) = \mathfrak{m}$. Let $\mathfrak{q}$ be any maximal ideal of $B_\mathfrak{p}$. We have an injective and finite map $A_\mathfrak{p}/\phi_\mathfrak{p}^{-1}(\mathfrak{q}) \to B_\mathfrak{p}/\mathfrak{q}$. By assumption, the ring $B_\mathfrak{p}/\mathfrak{q}$ is a field and by Lemma 5.13, the ring $A_\mathfrak{p}/\phi_\mathfrak{p}^{-1}(\mathfrak{q})$ is also field, ie $\phi_\mathfrak{p}^{-1}(\mathfrak{q})$ is a maximal ideal in $A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is a local ring, we have $\mathfrak{p} = \phi_\mathfrak{p}^{-1}(\mathfrak{q})$. □

Lemma 5.14. *Let $\phi : A \to B$ be an injective map of rings. Suppose that ϕ makes B into a finitely generated A-algebra. Suppose that A is a domain. Then the following two conditions are equivalent:*

(a) *B is a finite A-module;*

(b) *if $b_1, \ldots b_k$ is a set of generators of B as an A-algebra, there is for each i a monic polynomial $P_{b_i}(t) \in A[t]$ such that $P_{b_i}(b_i) = 0$.*

Proof. (a)⇒(b). Apply Lemma 5.11 to the multiplication by b map $(\bullet) \cdot b : B \to B$.

(b)⇒(a). Any element of B can be expressed as a polynomial in the b_i with coefficients in A. On the other hand, by assumption, for each b_i some power $b_i^{l_i}$ is a A-linear combination of powers b_i^l with $l < l_i$. Hence every element of B can be expressed as a linear combination of the elements

$$b_1, \ldots, b_1^{l_1-1}, b_2, \ldots, b_2^{l_2-1}, \ldots b_k, \ldots, b_k^{l_k-1}.$$

□

Lemma 5.15. *Let $\phi : A_0 \to B_0$ be an injective morphism of rings. Suppose that A_0 and B_0 are integral rings and suppose that B_0 is finitely generated as an A_0-algebra. Then there is $n \in \mathbb{N}$, $s \in A_0$ and a finite and injective homomorphism of A_0-algebras*

$$A_{0,s}[t_1, \ldots, t_n] \to B_{0,s}$$

Here $A_{0,s}$ is (as usual) the localisation of A_0 at the multiplicative set generated by s and $B_{0,s}$ is the localisation of B_0 at the multiplicative set generated by $\phi(s)$.

Proof. Let K be the fraction field of A_0. Consider the map

$$\phi_K : K \to B_0 \otimes_{A_0} K \simeq B_{0,\phi(A_0^*)}$$

given by the formula $\phi_K(a/b) = 1 \otimes a/b$. The map ϕ_K is also injective and $B_0 \otimes_{A_0} K$ is a domain and a finitely generated K-algebra. A similar reasoning was already made in the proof of Theorem 5.10 and we leave the details of the verification of these facts to the reader. By Noether's normalisation lemma 5.9, there is an injective and finite map of K-algebras

$$\lambda : K[t_1, \dots, t_n] \to B_{0,\phi(A_0^*)}$$

For $i \in \{1, \dots, n\}$, let $b_i/a_i := \lambda(t_i)$, where $b_i \in B_0$ and $a_i \in A_0^*$. Let s_0 be a multiple of $a_1 \cdot a_2 \cdots a_n$. Let

$$\lambda_0 : A_{0,s_0}[t_1, \dots, t_n] \to B_{0,\phi(s_0)}$$

be the morphism of A_{0,s_0}-algebras sending t_i to b_i/a_i. The map λ_0 is injective because A_0 is a domain and λ is injective. Let now $c_1/d_1, \dots c_k/d_k \in B_{0,\phi(s_0)}$ be a set of generators of $B_{0,\phi(s_0)}$ as a A_{0,s_0}-algebra. By Lemma 5.14, for each of the c_i/d_i there is a monic polynomial $P_i(t)$ with coefficients in $K[t_1, \dots, t_n]$, such that $P(c_i/d_i) = 0$ in $B_{0,\phi(A_0^*)}$. We may suppose wrog that the product of all the denominators appearing in the coefficients of all the P_i divides s_0. We conclude from Lemma 5.14 that $B_{0,\phi(s_0)}$ is a finite A_{0,s_0}-algebra. □

Proof. (of the theorem of Chevalley-Tarski).

Step I. We shall first prove the following statement. *If $f(E)$ is Zariski dense in Y then $f(E)$ contains a non empty open subset of Y.*

We shall prove this statement. First, by noetherian induction, we may assume that $\bar{E} = X$ and assume that the statement holds if $\bar{E} \neq X$.

Suppose that $f(E)$ is Zariski dense in Y. We may assume wrog that X is irreducible. To see this, suppose that X has several irreducible components $X_1, \dots X_r$. We then have

$$\overline{f(X_1 \cap E)} \cup \dots \cup \overline{f(X_r \cap E)} \supseteq \overline{f(E)}$$

and thus for one of the sets $\overline{f(X_i \cap E)}$ we must have $\overline{f(X_i \cap E)} = Y$. We may thus replace X by X_i and E by $E \cap X_i$. We may also assume that Y is irreducible (the argument is similar).

We may also suppose that X and Y are reduced, since the statement to be proven is topological.

My may wrog replace Y by one of its open affine subschemes V and X by $X \times_Y V$, so we may assume that Y is affine.

Let now $U \subseteq X$ be a non empty open affine subscheme. Either $f(U \cap E)$ is Zariski dense in Y or $f((X \backslash U) \cap E)$ is Zariski dense in Y. In the latter case, the assertion follows from the noetherian inductive hypothesis so we may assume that $f(U \cap E)$ is Zariski dense in Y and thus replace X by U. Now by definition, E is a finite union of locally closed subsets and one of these closed subsets, say E_0, must be dense in X. In particular, E_0 contain an affine open subset U_0 of X and as before, we may replace X by U_0 so that we now have $E = X$. By Lemma 5.2, we may thus assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, where A and B are integral rings. Let $\phi : A \to B$ be the corresponding maps of rings. We deduce from the fact that $f(X)$ is dense in Y that ϕ is injective (we ask the reader to prove this in Exercise 5.2).

So we are now reduced to show that that if $\phi : A \to B$ is an injective map of rings, which makes B a finitely generated A-algebra, then $\mathrm{Spec}(\phi)(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$ contains an open subset of $\mathrm{Spec}(A)$.

Now recall that by Lemma 5.15, there is $n \in \mathbb{N}$, $s \in A$ and a finite and injective homomorphism of A_0-algebras

$$A_s[t_1, \ldots, t_n] \to B_s$$

Here A_s is (as usual) the localisation of A at the multiplicative set generated by s and B_s is the localisation of B at the multiplicative set generated by $\phi(s)$. We may wrog replace A by A_s and B by B_s, since $\mathrm{Spec}(A_s)$ is a basic open subset of $\mathrm{Spec}(A)$. In this situation, Theorem 5.10 implies that $\mathrm{Spec}(\phi)$ is surjective and we have proven the statement and completed Step I.

Step II. By noetherian induction, we may assume that the Zariski closure of $f(E)$ is Y (use Exercise 3.3 and 4.7) and that the intersection of $f(E)$ with any proper closed subset of Y is constructible. Let $U \subseteq f(E)$ be the non empty open subset of Y whose existence is predicted by Step I. We know that $f(E) \cap (Y \backslash U)$ is constructible by the noetherian inductive hypothesis and thus $f(E)$ is the union of two constructible subsets of Y and is thus constructible. □

5.4 exercises

Exercise 5.1. *Prove Lemma 5.4.*

Exercise 5.2. *Let $\phi : A \to B$ be a morphism of integral rings. Suppose that $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has dense image. Show that ϕ is injective.*

Exercise 5.3. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. Let $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) \hookrightarrow (X, \mathcal{O}_X)$ be the closed reduced subscheme of $(X, \mathcal{O}_X)$ associated with the closed subset X. Show that $(X, \mathcal{O}_X)$ is affine if and only if $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$ is affine.*

Exercise 5.4. *Let $f : X \to Y$ be a strongly projective morphism. Suppose that Y is noetherian. Let F be a coherent sheaf over X and suppose that F is flat over Y. Suppose also that for some $k \geqslant 0$ and some $y \in Y$ we have $H^k(X_{\kappa(y)}, F|_{X_{\kappa(y)}}) = 0$. Prove that the coherent sheaf $R^k f_*(F)$ vanishes in a neighbourhood of y.*

Here are some explanations on the notations. Let $X_y := X \times_{\mathrm{Spec}(\kappa(y))} Y$ and let $\iota : X_y \to X$ be the first projection. We let $F|_{X_{\kappa(y)}} := \iota^*(F)$.

Exercise 5.5. *Let X be a noetherian scheme and let L, M be line bundles on X.*

- *Suppose that L is ample. Show that for sufficiently large $n \geqslant 0$, the line bundle $L^{\otimes n} \otimes M$ is ample.*
- *Suppose that L and M are ample. Show that the line bundle $L \otimes M$ is ample.*